Hartree-Fock Approach for Rotating Bose-Einstein Condensation in One-dimensional Deep Optical Lattice

Azza M. El-Badry, S. S. M. Soliman and M. M. Salah
Department of Physics, Faculty of Science, Minia University, El Minia, Egypt

IN THIS work, we investigated the thermodynamic behavior of a rotating Bose-Einstein condensation with non-zero interatomic interactions in one-dimensional optical lattice theoretically. Our system is formed by loading three dimensional boson-clouds into 1D optical lattice and subjected to rotate with angular velocity ꞷ about the z axis. We employed the semiclassical Hartree-Fock approach to calculate the condensate fraction, critical temperature, entropy and the heat capacity of the system. Thereby, we investigated the effect of the rotation rate, optical potential depth and the interactions parameter on these parameters. The obtained results provide a solid theoretical foundation for the current experiments of rotating interacting Bose-Einstein condensation produced or transferred in optical lattices.

Introduction

Bose-Einstein condensation (BEC) rotating in one dimensional (1D) optical lattice is a promising tool to simulate various problems from condensed matter physics [1-5]. In particular, BEC in 1D optical lattice constitutes a multilayer system, which has a great potential of applications. This system has an intriguingly similar layered structure such as the high-T_c superconductors in a magnetic field. As well as, it can be used to investigate the excitations, stability and dynamics of a vortex line in superfluid [6]. Moreover, this system can be used to reach fractional quantum Hall state [7]. However, this wide range of applications makes the inclusion of interatomic interaction an essential requirement for the correct description of the system.

In our previous work [8], we used the semiclassical Hartree-Fock approximation to investigate the interaction effect on the rotating condensate boson gas. In the present work, the relevant thermodynamic parameters of rotating BEC in a 1D deep optical lattice under realistic experimental condition will be calculated. These parameters enable us to illustrate the existing experimental data from a theoretical viewpoint. However, the quantum statistical mechanics of the interacting system remain unsolvable and one has to resort to approximated schemes. In this respect, the semiclassical Hartree-Fock (HF) approximation [9] provides the scheme mostly used for taking into account the interatomic interactions [10]. This mean-field theory avoids the difficulty of solving the full many-body Schrödinger equation for an interacting system by reducing the many-body problem to a one-body problem via the introduction of an appropriate mean field potential generated by all the other particles.

Motivated by the recent advances achieved in the experimental manipulation of rotating condensates [11,12] and of condensates loaded into optical lattices[13], we use the self-consistent HF to study the temperature dependence of the thermodynamic parameters for the rotating interacting BEC in 1D optical lattice. The effects of the rotation rate and optical potential depth are considered simultaneously. Many open questions related to exploring the effects of interatomic interactions, rotation rate and optical potential depth on the behavior of this system under different circumstances [14] are considered. These include: the BEC transition temperature [15,16]; the heat capacity, which enabled us to discuss the order of phase transition [17,18] and the entropy of the system[19], required to investigate the adiabatic cooling of the boson system in lattice. Our results are given for the trap parameters of the Hadzibabic's *et al.* experiment [13]: the radial and axial frequencies of the harmonic trap are $\omega_\perp$

$= 2\pi \times 74$ Hz, $\omega_z = 2\pi \times 4$ kHz and $\omega_z = 2\pi \times 80$ Hz. The optical potential depth normalized to the recoil energy $s = E_R/V_0$ ranging from 0 ~100 and the lattice spacing $d_z = 2.7\mu m$. The calculated results showed that the thermodynamic properties depend strongly on the simultaneous effect for rotation rate and the optical potential depth for the whole temperature range.

The paper is planned as follows: section two includes the system definition. The self-consistent Hartree-Fock model for the system under consideration is given in section three. The thermodynamic quantities are given in section four. Discussion and conclusion are given in the last section.

Basic formalism

We consider a mesoscopic sample of weakly interacting N bosonic atoms of mass m placed in an axially symmetric harmonic potential combined with one-dimensional optical potential in the radial direction,

$$V(r_\perp, z) = V_{har}(r_\perp, z) + V_{lat}(z)$$

$$= \frac{1}{2}m(\omega_\perp^2 r_\perp^2 + \omega_z^2 z^2) + V_0\left[sin^2\left(\frac{\pi z}{d_z}\right)\right] \quad (1)$$

with $r_\perp^2 = x^2 + y^2$ is the perpendicular radius and $[\omega_\perp \equiv \omega_x = \omega_y, \omega_z]$ are the effective trapping frequencies of the harmonic potential, V_0 is the optical potential depth and d_z is the lattice spacing in a direction z.

For the potential (1), it is impossible to find an exact analytical expression for the energy eigenvalues. However, an approximated expression can be readily obtained by considering an accurate approximation for the optical lattice potential V_{lat}. For a very deep lattice, where the atoms are localized at the potential minima of the optical lattice and hopping between different lattice sites is negligible, the optical potential can be safely approximated by an equivalent harmonic potential [2],

$$V_{lat} = \frac{1}{2}m\,\omega_{lat}^2\,z^2$$

with on-site trapping frequency, $\omega_{lat}^2 = \frac{4\,s\,E_R^2}{\hbar^2}, s$ is a dimensionless parameter and $E_R = \hbar\,\omega_R = \frac{\pi^2\,\hbar^2}{2md^2}$ is an energy scale for specifying the lattice depth. It is defined as the recoil energy that one atom requires when it absorbs one lattice photon.

Finally, the trapping potential is given by:

$$V_{trap}(r_\perp, z) = \frac{1}{2}m\omega_\perp^2 r_\perp^2 + \frac{1}{2}m\,(\,\omega_z^2 + \omega_{lat}^2) \quad (2)$$

The harmonic approximation is valid provided that the well is deep enough to contain several bound states and that the temperature is low enough that only states at the bottom of the well are occupied. In the rotating frame, this stirring potential reads [20,21],

$$V_{rot}(r_\perp, z) = \frac{1}{2}m[\omega_\perp^2(1 - \alpha^2)r_\perp^2 + (\omega_z^2 + \omega_{lat}^2)\,z^2] \quad (3)$$

where, $\alpha = \frac{\Omega}{\omega_\perp}$, is the rotation rate and Ω is rotation frequency around the z axis.

The Hamiltonian, describing the interacting atomic gas in the potential (3) is given by Cooper [22].

$$H = \frac{|P_\perp - m\Omega\times r_\perp^2|^2}{2m} + \frac{P_z^2}{2m} + V_{eff}(r_\perp, z) \quad (4)$$

where $V_{eff}(r_\perp, z)$ is the most effective potential for rotating interacting boson in optical lattice,

$$V_{eff}(r_\perp, z) = V_{rot}(r_\perp, z) + 2g[n_{th}(r_\perp, z) \cdot \quad (5)$$

with $g = \frac{4\pi\,\hbar^2 a}{m}$, is the interaction strength, $n_0(r_\perp, z)$ and $n_{th}(r_\perp, z)$ are the density of condensate and thermal atoms in the rotating frame.

Hartree-Fock approximation

In the self-consistent Hartree-Fock model, the condensate part satisfies the time-independent Gross-Pitaevskii equation (GPE)

$$[\frac{|p_\perp - m\Omega\times r_\perp^2|^2}{2m} + \frac{p_z^2}{2m} + V_{rot}(r_\perp, z) + g\,n_0(r_\perp, z)$$

$$+2gn_{th}(r_\perp, z)]\,\varphi(r_\perp, z) = \mu\varphi(r_\perp, z) \quad (6)$$

The GPE determines the condensate density quantum mechanically, $n_0 = |\varphi(r_\perp, z)|^2$ whereas the thermal atoms are treated semiclassically as a locally homogeneous ideal gas of bosons. The effect of the rotation is to change the shape of the distribution function of the thermal atoms so that, the thermal density out of the condensate takes the form [23,24],

$$N_T = \iint n_{th}(p, r)\,d^2p_\perp dp_\perp\,d^2r_\perp dz$$

$$= \iint \frac{a^- p_\perp a p_z\,a^- r_\perp a z}{(2\pi\hbar)^3}\,\frac{1}{e^{\beta(H - \mu)} - 1} \quad (7)$$

After substituting the Hamiltonian H and

doing the $\mathbf{p}$ integration by making the change of variables $\mathbf{p} \to \mathbf{p}_\perp - m\mathbf{\Omega} \times \mathbf{r}$ [25], the integral in Eq. (7) takes the same form as in the absence of rotation with an effective frequencies $\sqrt{\omega_\perp^2(1-\alpha^2)}$ and $\sqrt{(\omega_z^2 + \omega_{lat}^2)}$,

$$N_T = \frac{1}{\lambda_{th}^3} \sum_{j=1}^{\infty} \frac{z^j}{j^{3/2}} \int e^{-j\beta V_{eff}(r_\perp,z)} \, d^2 r_\perp dz \qquad (8)$$

where $\lambda_{th} = \sqrt{\frac{2\pi \, \hbar^2}{m k_B T}}$ is the thermal de-Broglie wavelength and $z = e^{\beta \mu}$ is the effective fugacity.

Equations (6) and (8), along with the constraint that the total number of atoms N is fixed

$$N = \int (n_{th}(r_\perp,z) + n_0(r_\perp,z)) \, d^2 r_\perp dz \qquad (9)$$

form a closed set of equations, define the Hartree-Fock theory for rotating condensate in optical lattice, which must be solved self-consistently.

Condensate density and chemical potential

Both the condensate density $n_0(r_\perp,z)$ and μ can be calculated from GPE for the condensate part, Eq.(6). The situation may be simplified by taking two advantages:

1. Firstly, one can neglect any influence of the thermal component on the spatial distribution of the condensate wave function

2. Secondly, we can neglect the mean-field energy due to the thermal component itself.

In this case, the condensate density, $n_0(r_\perp,z)$, can be obtained through Thomas-Fermi approximation (the kinetic energy term is omitted) of Eq6), *i.e.*

$$n_0(r_\perp,z) = |\varphi(r_\perp,z)|^2 = \frac{\mu - V_{rot}(r_\perp,z)}{g}$$

$$= \frac{1}{2} m[\omega_\perp^2 (1-\alpha^2)r_\perp^2 + (\omega_z^2 + \omega_{lat}^2) z^2] = \frac{\mu}{g}$$

$$[1 - \frac{\omega_\perp^2}{R_\perp^2(\alpha)} - \frac{z^2}{R_z^2(\alpha)}] \qquad (10)$$

For all $\mu > V_{rot}(r_\perp,z)$, $n_0(r_\perp,z) = 0$ elsewhere. In Eq. (10), the parameters $R_\perp(\alpha)$ and $R_z(\alpha)$ are the Thomas-Fermi radii at which the condensate density drops to zero along $r_\perp$ or z axis, .These radii are given by:

$$R_\perp(\alpha) = \sqrt{\frac{2\mu}{m \, \omega_\perp^2 (1-\alpha^2)}} \quad \text{and} \quad R_z(\alpha) = \sqrt{\frac{2\mu}{m(\omega_z^2 + \omega_{lat}^2)}} \qquad (11)$$

Thus, both of them accounted for the condensate radius in terms of the trap parameters and can be expressed in terms of the condensate number of atoms through the relation between μ and N_0 The relation between μ and N_0 can be found by integrating (10) over the ellipsoid with semi-axes $R_\perp$ and R_z,

$$N_0 = \int n_0(r_\perp,z) \, d^2 r_\perp dz = \frac{8\pi}{15} \frac{\mu}{g} R_\perp^2(\alpha) \qquad (12)$$

Using Eq.(11) in Eq.(12), one has

$$\mu = \frac{1}{2} \hbar\omega_g \left(\frac{15 N_0 a}{a_{har}}\right)^{2/5} (1-\alpha^2)^{2/5} (1 + \frac{\omega_{lat}^2}{\omega_z^2})$$

$$= \mu(0) \, (1-\alpha^2)^{2/5} (1 + \frac{\omega_{lat}^2}{\omega_z^2}) \qquad (13)$$

where $\mu(0) = \frac{1}{2} \hbar\omega_g \left(\frac{15 N_0 a}{a_{har}}\right)^{2/5}$ is the chemical potential for non rotating condensate, a is the s-wave scattering length, $a_{har} = \sqrt{\frac{\hbar}{\omega_g}}$ and $\omega_g = (\omega_\perp^2 \, \omega_z)^{1/3}$. The parameterized μ in Eq.(13) looks like a generalization for the well known Thomas-Fermi approximation of rotating condensate in deep optical lattice. Moreover, within the same approximation the effective potential is given by:

$$V_{eff}(r_\perp,z) = V_{rot}(r_\perp,z) + 2gn_0(r_\perp,z) =$$

$$|V_{rot}(r_\perp,z) - \mu| + \mu \qquad (14)$$

Thus, μ is the relevant energy scale parameterizing the effects of interactions, up to the point in the trap where $\mu = |V_{rot}(r_\perp,z)|$ as pointed by Hadzibabic and co-worker [14]. Following their approach which considered that (compared with μ/k_B) the majority of thermal atoms lie outside the condensate in the region where $V_{eff}(r_\perp,z) > \mu$ and $V_{eff}(r_\perp,z) = V_{rot}(r_\perp,z)$, the thermal atoms can be approximated by:

$$N_T = \frac{1}{\lambda_{th}^3} \sum_{j=1}^{\infty} \frac{z^j}{j^{3/2}} \int e^{-j\beta V_{rot}(r_\perp,z)} \, d^2 r_\perp dz$$

$$= \frac{1}{\lambda_{th}^3} \sum_{j=1}^{\infty} \frac{e^{-j\beta\mu}}{j^{3/2}} \int e^{-j\beta[\frac{1}{2}m[\omega_\perp^2(1-\alpha^2)r_\perp^2 + \omega_z^2\left(1+\frac{\omega_{lat}^2}{\omega_z^2}\right)z^2]} \, d^2 r_\perp dz \quad (15)$$

In terms of the thermal radii, which is equivalent to the Thomas-Fermi radii given in Eq.(11) and fixed the maximum value of the chemical potential compared to $k_B T$ and drops to zero along $T \to 0$.

$$R_\perp'(T) = \sqrt{\frac{2k_B T}{m \, \omega_\perp^2 (1-\alpha^2)}} \quad \text{and} \quad R_z'(T) = \sqrt{\frac{2k_B T}{m \, \omega_z^2(1+\omega_{lat}^2/\omega_z^2)}} \qquad (16)$$

The thermal atoms are given by:

$$N_T = \frac{1}{\lambda_{th}^3} \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \int e^{-j\left(\frac{r_\perp^2}{R_\perp^2} + \frac{z^2}{R_z^2} - \alpha_0\right)} d^2r_\perp dz$$

$$= 4\pi \frac{R_\perp'^2 R_z'}{\lambda_{th}^3} \sum_{j=1}^{\infty} \frac{1}{j^{3/2}} \int_{\sqrt{\alpha_0}}^{\infty} R^2 e^{-j(R^2 - \alpha_0)} dR \quad (17)$$

Following our procedure illustrated in our previous work [8] we have,

$$N_T = \frac{1}{\gamma} \left(\frac{k_B T}{\hbar\omega_g}\right)^3 (\zeta(3) + \alpha_0\zeta(2)) \quad (18)$$

with

$$\gamma = \sqrt{(1-\alpha^2)^2\left(1 + \frac{\omega_{lat}^2}{\omega_z^2}\right)} = \sqrt{(1-\alpha^2)^2\left(1 + 4s\frac{\omega_{lR}^2}{\omega_z^2}\right)}$$

$$\alpha_0 = \frac{\mu}{k_B T} = \eta \left(1 - \left(\frac{T}{T_0}\right)^3\right)^{2/5} \left(\frac{T_0}{T}\right) \gamma^{2/5} \quad (19)$$

and $T_0 = \frac{\hbar\omega_g}{k_B} \left(\frac{N}{\zeta(3)}\right)^{1/3}$ is the BEC transition temperature of a harmonically trapped non-rotating atoms. The parameter η in Eq.(19), first introduced by Stringari et al. [9,25], is determined by the ratio between the chemical potential at $T = 0$ value calculated in Thomas-Fermi approximation and T_0 in the same trap, i.e.

$$\frac{\mu(0)}{k_B T_0} = \eta \left(1 - \left(\frac{T}{T_0}\right)^3\right)^{2/5}$$ (the typical values for η of most experiments ranges from 0.3 to 0.4.). Finally the total number of particles is given by

$$N = N_0 + \frac{1}{\gamma} \left(\frac{k_B T}{\hbar\omega_g}\right)^3 (\zeta(3) + \alpha_0\zeta(2)) \quad (20)$$

Similarly, using the same procedure, one can also obtain results for the total energy E [9],

$$E(T) = \iint \frac{d^2p_\perp dp_z\, d^2r_\perp dz}{(2\pi\hbar)^3} \frac{H}{e^{\beta(H-\mu)} - 1} \quad (21)$$

and the local grand potential,

$$q = q_0 + q_{th} = \frac{1}{\lambda_\perp^3} \sum_{j=1}^{\infty} \frac{z^j}{j^{5/2}} \int e^{-j\beta V_{eff}(r_\perp, z)} d^2r_\perp dz$$

$$= E_0 + \frac{3k_B T}{\gamma} \left(\frac{k_B T}{\hbar\omega_g}\right)^3 (\zeta(4) + \alpha_0\zeta(3))$$

$$= q_0 + \frac{1}{\gamma} \left(\frac{k_B T}{\hbar\omega_g}\right)^3 (\zeta(4) + \alpha_0\zeta(3)) \quad (22)$$

Thermodynamic parameters
  Condensate fraction and critical temperature
  Using Eq.(20) for the total atoms number, the condensate fraction is given by:

$$\frac{N_0}{N} = 1 - \frac{1}{\gamma}\left\{\mathcal{T}^3 - \eta(1 - \mathcal{T}^3)^{2/5}\gamma^{2/5}\left(\frac{\zeta(2)}{\zeta(3)}\right)\mathcal{T}^2\right\} \quad (23)$$

with $\mathcal{T} = \frac{T}{T_0}$ which is the normalized temperature. In Eq.(23), the first term provides the condensate fraction in the thermodynamic limit. The second term, which vanishes for $\mathcal{T} > 1$ providing a consistent way for treating the interaction effect [26-29].

In the following, the calculated results will be considered for the experimental trap parameters of Hadzibabic [13]. In Fig.1, the dependence of the condensate fraction on $\mathcal{T}$ and α for optical potential depth $s = 50$ and interaction parameter $\eta = 0.4$ is illustrated. This figure shows that the condensate fraction has a monotonically decreasing nature for all α range 0,1}, the decreasing rate is minor in intermediate rotation regime and rapid in the fast rotation rate. The remarkable feature here is that the centrifugal force due to fast rotation exactly cancels the confining effect of the combined harmonic-lattice potential. Consequently, the condensate fraction drops to zero very quickly at α =1.

In Fig. 2 the dependence of the condensate fraction on $\mathcal{T}$ and s for $\alpha = 0.7$ and $\eta = 0.4$ is considered. This figure shows that the condensate fraction is independent on s for the deep optical potential depth. This behavior is different from the one for shallow optical potential depth [16].

The issue of the interaction effect on the condensate fraction for lattice depth $s = 50$ and the rotation rate $\alpha = 0.7$ are given in figure 3. This figure shows that the interaction effect leads to about 15% reduction in the condensate fraction. Finally, for deep optical lattice, the dependence of the condensate fraction on lattice depth is minor while on the interatomic interaction and the rotation rate α is considerable. Thus both two effects must be taken into consideration for a safe estimation of the critical rotating frequency (rotating frequency required to achieve the vortex state) and critical temperature.

The second term in Eq. (23) leads to a reduction of the condensate fraction. So, it is important to consider the effect of the interatomic interaction and the rotation rate on the transition temperature. This effect can be seen more clearly by calculating the critical temperature T_c. The latter is obtained as usual [17,30,31] by setting $\frac{N_0}{N}$ in Eq.(23) equal to zero, thus

$$T_c = T_0 \left\{1 - \frac{1}{3}\eta\gamma^{2/5}\left(\frac{\zeta(2)}{\zeta(3)}\right)\right\} \quad (24)$$

This result enabled us to investigate the effects of the η and alpha on $T_c(\Omega)$. Indeed, in Fig. 4, the normalized critical temperature $T_c(\Omega)/T_0$ is represented graphically as a function of rotation rate α and interaction effect η. This figure shows
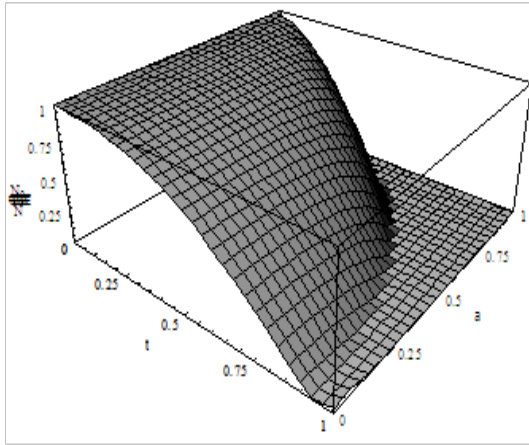
Fig.1: Condensate fraction versus the reduced temperature $\mathcal{T}$ and rotation rate α for $\eta = 0.4$ and s =50.
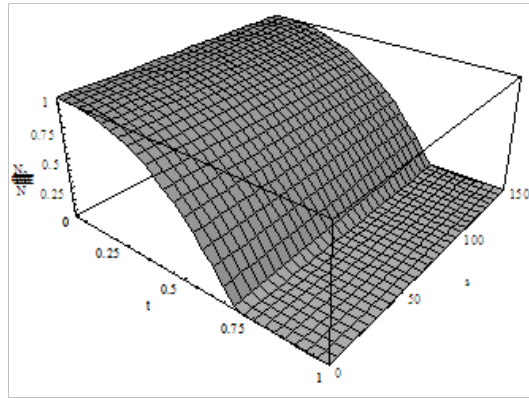

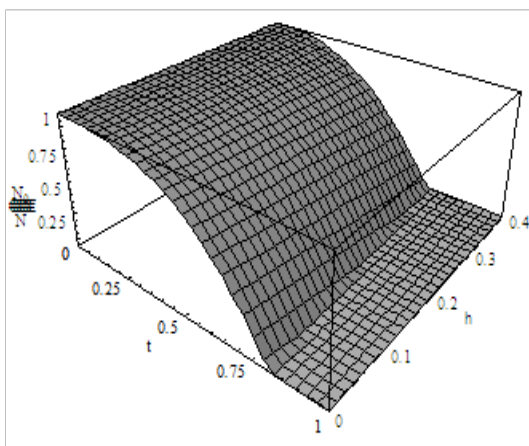Fig. 2. Condensate fraction versus the reduced temperature $\mathcal{T}$ and the lattice depth s for $\eta = 0.4$ and $\alpha = 0.7$.

Fig. 3. Condensate fraction versus the reduced temperature $\mathcal{T}$ and interaction parameter η for $\alpha = 0.7$ and s=50.

that the critical temperature T_c decreases as compared with the non-interacting case due to the repulsive nature of the interaction. The most remarkable point here is that the rate of T_c, decreasing for low rotation rate, is more rapid than the one of fast rotation rate.

Entropy of the system

A major goal in the field of degenerate quantum gases is to reach a suitable very low temperature. Such low temperatures are necessary
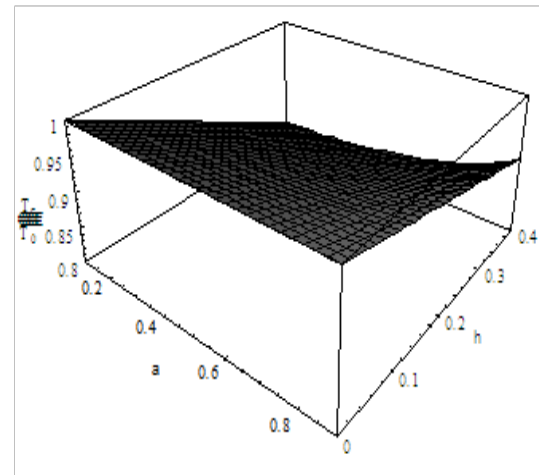

Fig. 4. Critical temperature T_c scaled by the non-rotating transition temperature T_0 for an interacting system, as a function of rotation rates α and interaction effect () for s=60   .

to reach phases relevant to condensed matter physics, such as quantum magnetism. However, it has been pointed out that loading sufficiently cold interacting boson atoms into a static optical lattice and is ramping the rotational frequency can lead to adiabatic cooling. To ascertain whether a given state is accessible, it is convenient to focus on its entropy rather than temperature. Thus, it is important to determine and investigate the entropy-temperature curves[32]. The behavior of these curves is used in analyzing the process of adiabatic cooling[19,33,34]. For the rotating condensate in an optical lattice, the normalized entropy per particle is given by Blakie [32]

$$\frac{S}{Nk_B} = \frac{q}{N} + \frac{E}{Nk_BT} - \frac{\mu}{k_BT}$$
$$= \frac{S_0}{Nk_B} - \frac{4}{\gamma}\frac{\zeta(4)}{\zeta(3)}\mathcal{T}^3 + \eta\left(1 - \mathcal{T}^3\right)^{2/5}\gamma^{2/5}\left\{\frac{4}{\gamma}\mathcal{T}^2 - \frac{1}{\mathcal{T}}\right\} \quad (25)$$

In Fig. 5, the entropy versus temperature curves as a function of α and s are given. These curves show that the entropy is always

fixed for temperatures sufficiently far below the transition temperature $\mathcal{T} < 0.4$. As the temperature increases, $\mathcal{T} \geq 0.5$, the entropy has a monotonically increasing nature everywhere.

Heat capacity

One of the sensitive quantities to clear up the effects of the rotation rate, optical potential depth and the interatomic interaction on the condensate is the behavior of the heat capacity as a function of the reduced temperature. In our approach, the heat capacity per a particle at constant volume $\frac{C_V(\mathrm{T})}{N k_B}$ is given by,
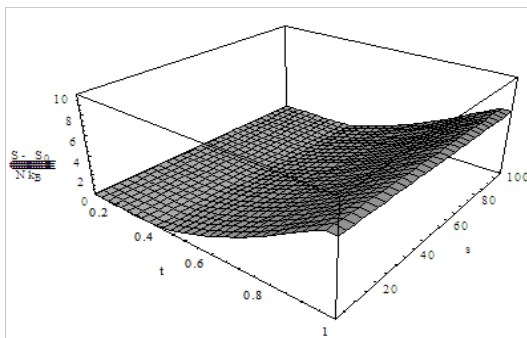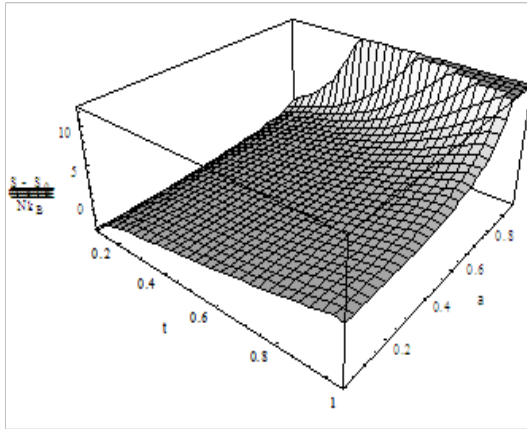
$$C_V(\mathrm{T}) = \left(\frac{\partial E}{\partial T}\right)_{N,V}. \quad (26)$$

However, it is known that for a given number of atoms, $C_V(\mathrm{T})$ increases to a maximum, then, falls rapidly to a saturation value as T increases greater than T_0. In such a situation, we must take into consideration two different temperature regimes, which are T less or greater than T_0.

For $\mathcal{T} < 1$, the heat capacity is given by:

$$\frac{C_{V,\mathcal{T}<1}}{Nk_B} = \frac{12}{\gamma}\mathcal{T}^3\left\{\frac{\zeta(4)}{\zeta(3)} - \eta\left(1 - \mathcal{T}^3\right)^{\frac{2}{5}}\left(\frac{1}{\mathcal{T}}\right)\gamma^{2/5}\right\}. \quad (27)$$

Fig. 5: Entropy versus normalized temperature $\mathcal{T}$ with α and s play as parameters.

While the heat capacity above the transition temperature, *i.e.* $\mathcal{T} > 1$, is given by:

$$\frac{C_{V,\mathcal{T}>1}}{Nk_B} = \frac{3}{\gamma}\mathcal{T}^3\left\{\left[4\frac{g_4(z)}{\zeta(3)} + \alpha_0\frac{g_3(z)}{\zeta(2)}\right] - \left[\frac{g_3(z)}{\zeta(3)} + \alpha_0\frac{g_2(z)}{\zeta(3)}\right] \times 3\left[\frac{g_3(z) + \frac{\alpha_0 g_2(z)}{\mathcal{T}}}{g_2(z) + \frac{\alpha_0 g_1(z)}{\mathcal{T}}}\right]\right\} (28)$$

in this case the total energy E is considered to be,

$$E(T) = E_0 + \frac{3k_BT}{\gamma}\left(\frac{k_BT}{\hbar\omega_g}\right)^3(g_4(z) + \alpha_0 g_3(z))$$

We now come to our analysis for the heat capacity using Eq's.(27) and (28). Eq.(27) shows that $\frac{C_{V,\mathcal{T}<1}}{Nk_B} \to 0$ when $\mathcal{T} \to 0$, *i.e.*

$$lim_{\mathcal{T}\to 0}\frac{C_V^\infty}{Nk_B} = 0 \quad (29)$$

thus, heat capacity (Eq.(29) obeys the third law of thermodynamics, which demands a vanishing heat capacity at zero temperature. For $< 0 < \mathcal{T} < 1$,the heat capacity is a smooth increasing function of $\mathcal{T}$ attaining its maximum value at temperature $\mathcal{T} = 1$. For $\mathcal{T} > 1$ the heat capacity drops suddenly to its asymptotic value. At $\mathcal{T} = 1$, the heat capacity becomes discontinuous. The magnitude of the jump is quite significant,

$$\frac{\Delta C_V}{Nk_B}\Big|_{\mathcal{T}=1} = 9\,\frac{\zeta(3)}{\zeta(2)} \quad (30)$$

this result reveals that the heat capacity of the rotating harmonically trapped boson in a deep optical lattice is equivalent to the heat capacity of harmonically trapped ideal bosons at the onset of condensation. The system under consideration is equivalent to the 3D harmonic oscillator with frequencies $\omega_\perp \pm \Omega$ and $\sqrt{(\omega_z^2 + \omega_{lat}^2)}$.

The Dulong-Petit law for our system can be extracted from Eq.(27). In the thermodynamic limit, as $\mathcal{T} \to \infty$ the asymptotic value of the heat capacity is given by:

$$lim_{\mathcal{T}\to\infty}\frac{C_V^\infty}{Nk_B} = 3\gamma \quad (31)$$

in calculating (31) the asymptotic expansion of $g_\nu(z)$ is used, for all value of ν the function $g_\nu(z)$ behaves like $z\left(\equiv 1 + O\left(\frac{1}{\mathcal{T}}\right)\right)$ itself.

The results calculated from Eq's. (27) and (28) are represented in Fig. 6 and 7 for different values of α and η , respectively. The approximation used in [35] is considered here to calculate Bose function $g_1(z)$ in Eq.(28). In Fig.6, we plot the heat capacity, as a function of the reduced temperature, and the rotation rate as well as the heat capacity

as a function of the reduced temperature and lattice depth are illustrated in Fig. 7. The heat capacity evolves, the heat capacity increasing smoothly with temperature for $\mathcal{T} < 1$, attaining its maximum values at the transition temperature, *i.e.* $\mathcal{T} = 1$, and then, decreases rapidly with increases the temperature $\mathcal{T} > 1$. It is continuous at the transition temperature, moreover its sharp peaks reminiscent to **λ**-shape. However, there is a discontinuity in the slope of the heat capacity, $\partial C_V / \partial \mathcal{T}$.
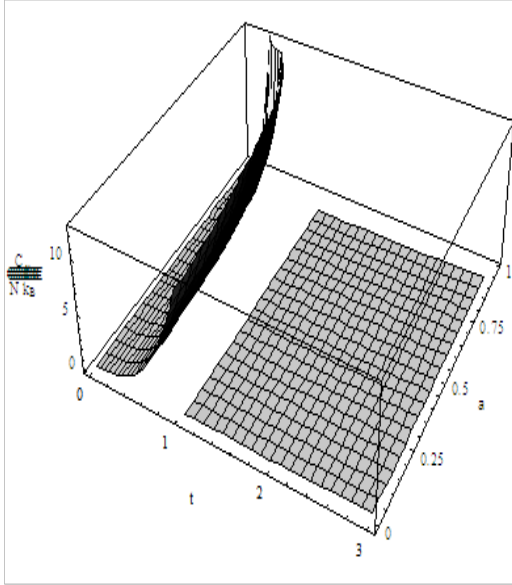

Fig. 6: Variation of the heat capacity $\dfrac{C_V(T)}{N\,k_B}$ with the reduced temperature $\mathcal{T}$ with α plays as a parameter.
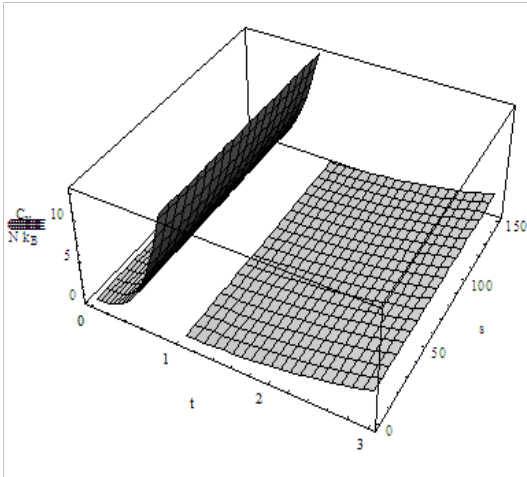

Fig.7: Variation of the heat capacity $\dfrac{C_V(T)}{N\,k_B}$ with the reduced temperature $\mathcal{T}$ with s plays as a parameter, for $\eta = 0.4..$

Discussion and Conclusion

In this paper, using the semiclassical Hartree-Fock approximation, we obtained an analytical expression for the thermodynamic parameters of a rotating interacting Bose gas in one dimensional deep optical lattice. Expressions for the condensate fraction: transition temperature, entropy and the specific heat are derived. Our approach provides the possibility to test the applicability of experimental realization of this system.

The calculated results showed that these thermodynamic quantities depend on the rotation rate, the optical potential depth as well as the interatomic interaction for all temperature range. The critical temperature and the condensate fraction are decreasing compared with the Bose rotating condensate gas case in the absence of the optical lattice. Using C_V as the indicator, we also investigated the phase transition from the gas phase to condensed phase.

First of all, one must bear in mind that our results are based on the interacting Bose gas model. As the rotation frequency increases from the slow rotation, there exists a dynamically unstable region of rotating velocities, *i.e.* there exists a critical rotation frequency. However, rotation effect leads to a shift in the radial harmonic oscillator frequencies, but still fulfill the condition $\hbar \omega_\perp (1 \pm \alpha_c)$ with α_c be the critical rotation rate. The latter provides the criterion stability of the rotating condensate, it does not necessarily indicate the critical frequency for vortex nucleation. The corresponding thermodynamic rotation rate can be estimated using the relation [36],

$$\alpha_c \approx 1 - \frac{Na}{\sqrt{8\pi}\, d_z}$$

We will discuss the required conditions to reach a stable rotating Q2D briefly. The Q2D can be reached when the thermal energy $k_B T$ is less than the site spacing energy in the z direction, *i.e.*,

$$k_B T \ll \hbar \sqrt{(\omega_z^2 + \omega_{lat}^2)} \equiv \hbar \omega_z \sqrt{1 + 4\,s\,(\omega_R^2/\omega_z^2)}$$

When the temperature T is of the order of this spacing, *i.e.*

$$k_B T \approx \hbar \omega_z \sqrt{1 + 4\,s\,(\omega_R^2/\omega_z^2)}, \quad (32)$$

the effect of the harmonic frequency ω_z becomes pronounced and can be used to control the dimensionality cross-over of the system. We also note that, decreases ω_z (increases ω_R) leads to a

cross-over to the strictly rotating 2D Bose gases. In this case, this thermal energy corresponds to the experimental situation with small, but not completely negligible, thermal occupation of a few excited states of the tightly confining potential. Result in [32] also reveal that, the thermal stability of the Q2D system depends on both the lattice depth and the magnetic frequency along z. The critical rotation rate provides the criterion stability of the rotating condensate and it does not necessarily indicate the critical frequency for vortex nucleation.

It is useful to compare the chemical potential, μ for the rotating condensate bosons in one dimensional deep optical lattice with the combined oscillator-optical energy in the tightly confinement direction. For the Q2D, the chemical potential should fulfill the condition

$$\mu \ll \hbar\omega_z \sqrt{1 + 4\,s\,(\omega_R^2/\omega_z^2)}$$

thus, in order to reach a stable Q2D the condition

$$\mu(0) \ll \hbar\omega_z\,(1 + 4\,s\,(\omega_R^2/\omega_z^2)^{1/10}(1-\alpha^2)^{-2/5}$$

must be satisfied. Finally, the analytical calculated results reveal that the system in a Q2D regime obeys the 2D statistics but interacts in the same way as in a 3D system. This important note is revealed from Eq.(13).

References

1. Hadzibabic, Z., Krüger, P., Cheneau, M., Rath, S. P. and Dalibard, J., *New J. Phys.* **10** (2008). 045006; Hadzibabic, Z., Krü, P., Cheneau, M., Battelier, B., and Dalibard, J., *Nature,* **441,** 1118(2006).

2. Jaksch, D. and Zoller, P., *New J. Phys.,* **5** 56;(2003) Jaksch, D., Bruder, C., Cirac, J. I., Gardiner, C. W. and Zoller, P., *Phys. Rev. Lett.* **81,** 3108 (1998).

3. Greiner, M., Mandel, O., Esslinger,T., Ansch,T.W. H. and Bloch, I., *Nature,* **415,** 39 (2002).

4. Morsch, O. and Oberthaler, M., *Rev. Mod. Phys.* **78,** 179(2006).

5. Pedri, P., Pitaevskii, L., Stringari, S., Fort, C., Burger, S., Cataliotti, F. S., Maddaloni, P., Minardi, F. and Inguscio, M., *Phys. Rev. Lett.* **87** , 220401(2001).

6. Martikainen, J.P. and Stoof, H. T. C., *Phys. Rev. Lett.* **91,** 240403 (2003) ; *Phys. Rev.* A 69, 053617 (2004) ; Kevrekidis, P. G., Carretero-Gonz\'alez, R., Theocharis, G., Frantzeskakis, D. J. and Malomed, B. A., *J. Phys.* B: *At. Mol. Opt. Phys.* **36,** 3467 (2003) ; Bhattacherjee, A. B., Morsch,

7. Wilkin, N. K. and Gunn, J. M. F., *Phys. Rev. Lett.* **84,** 000006(2000).

8. Hassan, A. S., El-Badry, A. M. and Soliman, S. S. M., *Eur. Phys. J.* **D 70** Accepted for publication (2016).

9. Dalfovo, F., Giorgini, S., Pitaevskii, L.P. and Stringari, S., *Rev. of Mod. Phys.* **71,** 463 (1999).

10. Giorgini, S., Pitaevskii, L. P. and Stringari, S., *Journal of Low Temperature Physics,* **109** 309 (1997).

11. Rosenbusch, P., Bretin, V. and Dalibard, J., *Phys. Rev. Lett.* **89,** 200403 (2002).

12. Bretin,V., Stock, S., Seurin, Y. and Dalibard, J., *Phys. Rev. Lett.* **92,** 050403 (2004).

13. Hadzibabic, Z., Stock, S., Battelier, B., Bretin, V. and Dalibard, J., *Phys. Rev. Lett.* **93,** 180403(2004).

14. Tammuz, N., Thermodynamics of ultracold ^{39}K atomic Bose gases with tunable interactions (P*h.D. Thesis,* Cavendish Laboratory, University of Cambridge, UK, 2011).

15. Stringari, S., *Phys. Rev. Lett.* **82,** 4371 (1999) ; *Phys. Rev. Lett.* **86,** 4725 (2001)

16. Hassan, A. S. and El-Badry, A. M., *Eur. Phys. J.* **D 68** (2014) 76; Hassan, A. S., El-Badry, A. M. and Soliman, S. S. M., *Eur. Phys. J.* **D 64,** 465 (2011) ; Hassan, A. S., *Phys. Lett.* A 374, 2106 (2010) ; Hassan, **A. S.,** El-Badry, A. M. and Soliman, S. S. M., *Physica,* **B 405,** 4768(2010).

17. Van Druten, N. and Ketterle, W., *Phys. Lett. Lett.* **79,** 549(1997).

18. Shiozaki, R. F., Telles, G. D., Castilho, P., Poveda Cuevas, F. J., Muniz,S. R. , Roati, G., Romero-Rochin,V. and Bagnato, V.S., *Phys. Rev.* **A 90,** 043640 (2014).

19. Olf, R., Fang, F., Marti, G. E., MacRae, A. and Stamper-Kurn, D. M., *Nature Physics,* **11,** 720 (2015).

20. Bargi, S., Kavoulakis, G. M. and Reimann, S. M. , *Phys. Rev.* **A 73,** 033613 (2006).

21. Pitaevskii, L. and Stringari, S., *Bose-Einstein Condensation,* Clarendon Press. Oxford chapter **14**(2003).

22. Cooper, N. R., *Adv. Phys.* **57,** 539(2008).

23. Sinha, S., *Phys. Rev.* **A 58,** 3159 (1998).

O., and Arimondo, E., *J. Phys.* B: *At. Mol. Opt. Phys.* **37,** 2355 (2004) ; Reijnders,J. W. and Duine, R. A., *Phys. Rev. Lett.* **93,** 060401(2004)

24. Haugerud, H., Haugset, T. and Ravndal, F., *Phys. Lett.* **A 225,** 18 (1997).

25. Giorgini, S. and Pitaevskii, L.P. and Stringari, S., *Phys. Rev. Lett.* **78,** 3987(1997).

26. Campbell, R.L.D., Smith, R. P., Tammuz, N., Beattie, S., Moulder, S. and Hadzibabic, Z., *Phys. Rev.* **A 82,** 063611 (2010).

27. Tammuz, N. Smith, R. P., Campbell,R. L. D., Beattie, S., Moulder, S., Dalibard, J. and Hadzibabic, Z., *Phys. Rev. Lett.* **106,** 230401 (2011).

28. Smith, R. P., Campbell, R. L. D., Tammuz, N. and Hadzibabic, Z., *Phys. Rev. Lett.* **106,** 250403 (2011)

29. Smith, R.P., Tammuz, N., Campbell, R.L.D., Holzmann, M. and Hadzibabic, Z., *Phys. Rev. Lett.* **107,** 190403 (2011) .

30. Kirsten, K. and Toms, D. J., *Phys. Lett.* **A, 222** 148 (1996) ; Kirsten, K. and Toms, D. J., *Phys. Rev.* **A 54,** 4188 (1996) ; Kirsten, K. and Toms, D. J., *Phys. Lett.* **A 243,** (1998) ; Kirsten, K. and Toms, D. *J., Phys. Rev.* **E 59,** 158 (1999).

31. Grossmann, S. and Holthaus,M., *Phys. Lett. A* **208,** 188 (1995).

32. Blakie, P. B. and Porto, J. V., *Phys. Lett.* **A 69** 013603 (2004).

33. Catani, J., Barontini, G., Lamporesi, G., Rabatti,F., Thalhammer, G., Minardi, F., Stringari, S. and Inguscio, M., *Phys. Lett.* **A 103,** 140401 (2009).

34. Schachenmayer, J., Weld, D.M., Miyake,H., Siviloglou, G.A., Ketterle, W. and Daley, A. J. *Phys. Rev.* **A 92,** 041602 (2015)

35. Klünder, B. and Pelster, A., *Eur. Phys. J.* **B 68,** 457 (2009).

36. Linn, M. and Fetter, A.L., *Phys. Rev.* **A 60,**4910 (1999) .

تقريب هارترى – فوك الشبه كلاسيكى لتكثيف بوز- اينشتاين الدوار فى بعد واحد لشبيكة ضوئية عميقة

عزه محمد البدرى ، شيمى شعراوى محمد سليمان و محمود محمد صلاح
قسم الفيزياء – كلية العلوم – جامعة المنيا- المنيا - مصر .

تمثل الخواص الثرموديناميكية لتكثيف بوز اينشتاين الدوار فى شبيكة ضوئية فى اتجاه محور الدوران وفى وجود التفاعل احد اهم الانظمة الموازية لدراسة انظمة المواد شديدة الارتباط (condensed matter physics). تناول هذا البحث دراسة الخواص الثرموديناميكية لغاز بوزونى دوار فى شبيكة ضوئية عميقة. تم استخدام طريقة هارترى-فوك المستخدمة فى حساب البارامترات الثرموديناميكية (الجزء المتكثف و درجة حرارة التكثيف والانتروبى والسعة الحرارية) . تعتمد هذه الطريقة علي تقليص معادلة شرودنجر لعدد N من الجسيمات الي معادلة لجسم واحد وذلك باستنباط جهد فعال (تقريب الجهد المستخدم الي جهد فعال) وبينت الحسابات ان هذه البارامترات تعتمد كثيرا على معدل الدوران وكذلك عمق الجهد الضوئى. وعلى الرغم من احتفاظ البارمترات الثرموديناميكية بنفس الخواص من حيث رتبة الانتقال الطورى والتحميل المعزول حراريا (Adiabatic) فإن الجهد الضوئى يؤدى الى تقليص الجزء المتكثف وكذلك درجة الحراره الحرجه للانتقال الطوري. يفسر هذا التقليص بأن الجهد الضوئى يسبب تقيدا لحركة الانتقال للذرات المتكثفة بين الاركان البلوريه فى الشبيكة الضوئية (lattice sites) . يبين هذا السلوك ان مثل هذا الانظمة تحتاج الى درجات حرارة تبريد كبيرة مقارنة بمثيلاتها فى حالة غياب الجهد الضوئى. يتفق هذا السلوك تماما مع التجارب العملية الخاصة بالمواد شديدة الارتباط.