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Quantum-induced Stress-Energy Tensor in the Relativistic Regime

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Abstract

The generalized noncommutative Heisenberg algebra based on the generalized uncertainty principle imposes a minimal length uncertainty to quantum mechanics (QM). On the other hand, quantum-induced spacetime is suggested as an additional curvature on the relativistic eight-dimensional tangent bundle (phase-space), with a complimentary term combining reconciling principles of QM with General Relativity (GR) and comprising the minimal length discretization and the first-order derivatives of tangent covectors, the quantum-induced torsion-free metric tensor could be constructed. Accordingly, quantum-induced corrections imposed on the symmetric stress-energy tensor, the source of spacetime curvature, and the energy density associated with the electromagnetic and scalar Lagrangian are also suggested. Besides the classical version of the stress-energy tensor, the proposed quantization introduces additional Lagrangian densities and potentials together with coefficients depending on the metric tensor, tangent covector derivatives, and physical constants including the gravitational constant, Planck constant, speed of light, and Planck length. The vanishing covariant derivative of the quantum-induced stress-energy tensor confirms Einstein's GR and suggests that the corresponding continuity equation implies that the gravitational fields do work on the classical and quantum matter and vice versa. For vanishing tangent covector's first derivative and/or vanishing minimal length uncertainty, the classical GR and the undeformed (orthodox) QM are fully retained. Accordingly, the Einstein stress-energy tensor is also retrieved. Thus, we conclude that the suggested quantum-induced stress-energy tensor is, in principle, suitable for both quantum and classical field equations.

Keywords: modified gravity, minimal length scale, generalized uncertainty principle, general relativity, stress-energy tensor, deformed phase space.

I- Introduction

Even though special relativity and quantum mechanics have been unified for quite some time, there have been no successful attempts to reconcile general relativity and quantum mechanics over the last century. This script presents a revisit to new method inspired by the early efforts made by Born [1–4], and by Caianiello [5–8]. We also utilize the recent progress in quantum geometry [9–13] and noncommutative algebra [14–21]. In a recent work of the present authors [22], we have shown that the application of quantum-induced deformation is believed to imply the generalization of the Riemannian spacetime geometry that forms the basis of classical general relativity theory to an eight-dimensional spacetime fiber bundle. This expansion governs the alteration of the line element, metric tensor, Levi-Civita connection, and Riemann curvature tensor. In the present work, we primarily investigate the implications of such quantum-induced deformation on the classical GR Lagrangian and the stress-energy tensor.

Unlike Caianiello's technique [23], which we have already discussed in depth in [22], our method is clearly independent of any particle mass scale. It expressly manifests a minimal length scale, which, depending on the minimal length scale selected, can naturally be coupled to a maximal acceleration scale via some quantum deformation parameter and/or certain fundamental physical constants. Furthermore, our approach's metric and

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other geometrical objects are observer-dependent, in contrast to Brandt's approach [24]. Thus, defining the minimum length (or maximum acceleration) does not necessitate the use of any universal length scale or mass scale. To be more precise, in our method, we presume that, excluding phenomenological limitations, we do not need to determine the minimal length scale to be precisely the Planck scale.

For a source of spacetime curvature and energy density, we assume a symmetric stress-energy tensor. Moreover, in an earlier work [25] with the quantum-induced torsion-free metric tensor, we concluded that the vanishing covariant derivative, the continuity equation implies that the gravitational fields do work on the classical and quantum matter and vice versa and the non-gravitational-energy and momentum are no longer conserved.

The present paper is organized as follows. The formalism is outlined in section 2. Section 2.1 is devoted to the relativistic generalized uncertainty principle in eight-dimensional fiber bundle. The quantum-induced stress-energy tensor is introduced in section 2.2. The quantum-induced corrections to the Lagrangian of scalar and electromagnetic fields are given in section 2.3. Section 2.4 elaborates on the quantum-induced corrections to the stress-energy tensor with an electromagnetic Lagrangian. The symmetry properties and covariant derivative of the stress-energy tensor with an electromagnetic Lagrangian are outlined in sub-sections 2.4.1 and 2.4.2, respectively. The quantum-induced stress-energy tensor with a scalar Lagrangian and its symmetry property and covariant derivative are worked out in sub-sections 2.5.1, and 2.5.2, respectively. Section 3 is devoted to the summary and conclusion.

II. Formalism

A. Modified metric tensor

The notion of minimum length as a result of the anticipated additional fuzziness of spacetime structure brought about by gravitational impacts close to the fundamental scales of very high energy required to resolve very small distances, L arises from the fact that a minimal measurable length uncertainty is predicted in various theories of quantum gravity. Kempf's version of GUP [26] is the one we use in the present paper. It suggests

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}[1+\beta(\Delta p)^2+\beta\langle p\rangle^2], \qquad (1)$$

where $\langle p\rangle$ is the momentum expectation value, Δx and Δp, respectively, represent the length and momentum uncertainties. The GUP parameter, $\beta = \beta_0 G/(c^3\hbar)$, with β_0 being a dimensionless parameter that encapsulates the transition to GUP and captures the effects of gravity on HUP, one of the fundamental ideas of QM. β_0 is in the order of 1 according to different independent theoretical estimations [27]. However, there is still much work to be done to increase the current empirical bounds from new cosmological data of gravitational and non-gravitational origin [27, 28]. In the present paper, we assume that β_0 is left to be determined empirically.

Now, the well-known commutation relation between momentum and length operators is as follows:

$$[\hat{x},\hat{p}] = i\hbar(1+\beta\hat{p}^2). \qquad (2)$$

The minimum uncertainty of position Δx_{min} for the whole range of expectation values of momentum $\langle p\rangle$ is

$$\Delta x_{min}(\langle p\rangle) = \hbar\sqrt{\beta}\sqrt{1+\beta\langle p\rangle^2}. \qquad (3)$$

When $\langle p\rangle^2 = 0$, the absolute minimum uncertainty of position becomes

$$\Delta x_0 = \hbar\sqrt{\beta} \qquad (4)$$

The minimal measurable length scale, Δx_0, can be used to determine the minimum position uncertainty. This value represents the distance at which quantum effects of the gravitational interaction are anticipated to become important. At that point, the minimal measurable length becomes

$$\begin{aligned} L &= \Delta x_0 \\ &= \hbar\sqrt{\beta} \end{aligned} \qquad (5)$$

In the following, we adhere to the basic concepts of Caianiello's original model to incorporate quantum effects of the gravitational interaction on the spacetime geometry near the fundamental scale at which such effects are expected to become prominent [29–31]. This implies that a four-dimensional spacetime embedded as a hypersurface in an eight-dimensional manifold M_8 can explain the classical GR and thus the classical spacetime geometry. We presume that the quantum regime, where we expect to observe the quantum-induced effects at a minimal length scale, is included in the 4-velocity space. The eight dimensions x^A in the manifold M_8, which are the extended dimensions, are

$x^A = (x^\mu, (L/c)\dot{x}^\mu),$ (6)

where x^μ is the four spacetime dimensions (four-dimensional sub-manifold of the manifold M_8), $\dot{x}^\mu = \frac{dx^\mu}{ds}$ is the four-velocity, $A = 0,\ldots,7$, $\mu = 0,\ldots,3$, L is the minimal length. Here, L may be defined according to Kempf's GUP model as a minimal uncertainty of position, Eq. (5); $L = \Delta x_0 = \hbar\sqrt{\beta}$

[26], or one can assume the value of minimal length to be the Planck length, $L = l_p = \sqrt{(\hbar G/c^3)}$.

The geometrical embedding previously described will be viewed in the context of this paper's primary goal as merely a formal process to create a new metric $\tilde{g}_{\mu\nu}$ from a given metric $g_{\mu\nu}$. This new metric differs from $g_{\mu\nu}$ by a correction (deformation) factor, which will be elaborated below.

The deformed line element ($d\tilde{s}^2$) with metric (g_{AB}) in the eight-dimensional manifold M_8 [24, 32] is thereby given by,

$d\tilde{s}^2 = g_{AB}dx^A dx^B$ (7)

where g_{AB} is a result of the outer product as the following

$g_{AB} = g_{\mu\nu} \otimes g_{\mu\nu}.$

In Eq. (7), we substitute for dx^A, and dx^B by the differential form of Eq.(6)

$d\tilde{s}^2 = (1 + Lg_{\mu\nu}\frac{dx^\mu}{ds}\frac{d\dot{x}^\nu}{ds} + Lg_{\mu\nu}\frac{d\dot{x}^\mu}{ds}\frac{dx^\nu}{ds} + \quad + L^2 g_{\mu\nu}\frac{d\dot{x}^\mu}{ds}\frac{d\dot{x}^\nu}{ds})ds^2,$ (8)

where $c = 1$, $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ is the classical line element,

$d\tilde{s}^2 = ds^2 + L^2 g_{\mu\nu}\ddot{x}^\mu\ddot{x}^\nu ds^2,$ (9)

where $\ddot{x}^\mu = \frac{d\dot{x}^\mu}{ds}$ is the acceleration of the particle, μ, ν are dummy indices, and $\vec{\dot{x}}.\vec{\dot{x}} = -1$, then $\vec{\dot{x}}.\vec{\ddot{x}} = 0$,

$d\tilde{s}^2 = (1 + L^2\ddot{x}^2)ds^2$ (10)

where $\ddot{x}^2 = g_{\mu\nu}\ddot{x}^\mu\ddot{x}^\nu$. The deformed line element in four dimensions spacetime, as a projection from eight dimensions into four dimensions, will be

$d\tilde{s}^2 = \tilde{g}_{\mu\nu}dx^\mu dx^\nu,$ (11)

where $\tilde{g}_{\mu\nu}$ is the deformed (modified) metric tensor.

The deformed metric tensor $\tilde{g}_{\mu\nu}$ is the quantum-induced metric of the spacetime hypersurface embedded in the extended manifold M_8. The relation between the deformed metric tensor and the classical metric tensor will be obtained by equating Eqs. (10), and (11),

$\tilde{g}_{\mu\nu} = (1 + L^2\ddot{x}^2)g_{\mu\nu},$ (12)

where $\ddot{x}^2 = g_{\alpha\gamma}\ddot{x}^\alpha\ddot{x}^\gamma$, γ, α are dummy indices, μ, and ν are free indices. For flat spacetime,

$\tilde{\eta}_{\mu\nu} = (1 + L^2\ddot{x}^2)\eta_{\mu\nu}.$ (13)

The relation between the correction factor $(1 + L^2\ddot{x}^2)$ and GUP can be derived by substituting for L from Eq. (12) by Eq. (5),

$\tilde{g}_{\mu\nu} = (1 + \mathcal{T}\ddot{x}^2)g_{\mu\nu},$ (14)

where $\mathcal{T} = \hbar^2\beta$.

B. Quantum-mechanical aspects imposed on the stress-energy tensor

The full action of the theory of general relativity in curved spacetime consists of the Einstein-Hilbert action and the non-gravitational part of the Lagrangian density $\mathcal{L}_{\text{matter}}$; the matter field. This is done by substituting the ordinary derivative ∂ with the covariant derivatives ∇ and the Minkowski metric tensor $\eta_{\mu\nu}$ with the fundamental (metric) tensor $g_{\mu\nu}$.

$$S = \int \frac{c^4}{16\pi G}(R + \mathcal{L}_{\text{matter}})\sqrt{-g}\, d^4x, \quad (15)$$

where Ricci scalar is denoted by R. Assuming that the Lagrangian density is scalar, the Jacobian term $\sqrt{-g} = \sqrt{-\det(g_{\mu\nu})}$, which is obtained in section IIA, ensures action invariance under diffeomorphisms. Demanding vanishing variation of this action about the inverse metric tensor $g^{\mu\nu}$ yields the physical quantities, including the Hilbert stress-energy tensor. [33]

$$T_{\mu\nu} = \frac{-2}{\sqrt{-|g|}}\frac{\partial}{\partial g^{\mu\nu}}\left(\sqrt{-|g|}\mathcal{L}_{\text{matter}}\right) = -2\frac{\partial \mathcal{L}_{\text{matter}}}{\partial g^{\mu\nu}} + g_{\mu\nu}\mathcal{L}_{\text{matter}}. \quad (16)$$

where $|g| = |\det(g_{\mu\nu})|$. The matter field and the fundamental tensor are the sole features that characterize $T_{\mu\nu}$. It is possible to rewrite the stress-energy tensor, Eq. (16), using the quantum-induced metric tensor, Eq. (14).

$$\tilde{T}_{\mu\nu} = -2\frac{\partial \tilde{\mathcal{L}}_{\text{matter}}}{\partial \tilde{g}^{\mu\nu}} + \tilde{g}_{\mu\nu}\tilde{\mathcal{L}}_{\text{matter}}. \quad (17)$$

The quantum-induced version of the matter Lagrangian is derived in section IIC. For now, we start with that of the metric (fundamental) tensor

$$\delta\tilde{g}^{\mu\nu} = (1 + \mathcal{T}|\ddot{x}|^2)^{-1}\delta g^{\mu\nu}$$
$$-(1 + \mathcal{T}|\ddot{x}|^2)^{-2} g^{\mu\nu}\mathcal{T}\delta\left(g^{\mu\nu}\ddot{x}_\mu\ddot{x}_\nu\right).$$

For $\tilde{g}^{\mu\nu} = (1 + \mathcal{T}|\ddot{x}|^2)^{-1}g^{\mu\nu}$ and $|\ddot{x}|^2 = g^{\mu\nu}\ddot{x}_\mu\ddot{x}_\nu$,

$$\delta\tilde{g}^{\mu\nu} = (1 + \mathcal{T}|\ddot{x}|^2)^{-1}\delta g^{\mu\nu} - (1 + \mathcal{T}|\ddot{x}|^2)^{-2}\mathcal{T} \times \left[|\ddot{x}|^2\delta g^{\mu\nu} + g^{\mu\nu}\ddot{x}^\mu\delta\ddot{x}_\mu + g^{\mu\nu}\ddot{x}^\nu\delta\ddot{x}_\nu\right], \quad (18)$$

where $\ddot{x}^\mu = g^{\mu\nu}\ddot{x}_\nu$, $\ddot{x}^\nu = g^{\mu\nu}\ddot{x}_\mu$. Then, Eq. (18) can be substituted into Eq. (17). When replacing the small variation by differentiation and $\tilde{g}_{\mu\nu}$ by $g_{\mu\nu}$, then, the quantum-induced stress-energy tensor is given so far as

$$\tilde{T}_{\mu\nu} = \frac{-2(1 + \mathcal{T}|\ddot{x}|^2)^2}{1 - \mathcal{T}g^{\mu\nu}\left(\ddot{x}^\mu\frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu\frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)}\frac{\partial \tilde{\mathcal{L}}_M}{\partial g^{\mu\nu}}$$
$$+(1 + \mathcal{T}|\ddot{x}|^2)g_{\mu\nu}\tilde{\mathcal{L}}_M. \quad (19)$$

When comparing Eq. (19) with Eq. (17), we conclude that

- both common terms, Eq. (18), are entirely retrieved, and
- both of them are multiplied by coefficients that differ from unity.

These coefficients depend on quantum-mechanical quantities such as β_0, the classical metric tensor, and the variation of the first-order derivatives of tangent covectors $|\ddot{x}|^2$ with respect to the classical metric tensor. According to the quantum-induced correction suggested, finite β_0 and/or $|\ddot{x}|^2$, linearly factor Eq. (19). When this factor is removed, the classical stress-energy tensor, Eq. (17), is entirely restored. Thus, we conclude that the corrected stress-energy tensor suggested in Eq. (19) is valid in both classical and quantum regimes. We also conclude that by switching β_0 (and/or $|\ddot{x}|^2$) off or on, classical or quantum stress-energy tensor is fully constructed. Such a linear factorization suggests a generalization of the stress-energy tensor because of the complementary term added to the fundamental tensor, Eq. (14).

C. Quantum-induced matter Lagrangian density

The generalized Lagrangian density is expressed as [34],

$$\mathcal{L}(q^i, \dot{q}^i) = \frac{1}{2}\sum_{i,j=1}^{n} g_{ij}(q)\dot{q}^i\dot{q}^j + V(q^i), \quad (20)$$

where q^i are the generalized coordinates on manifold M_4 and $\dot{q}^i$ represent phase-space velocities on tangent bundle TM_4.

For any type of matter field $\psi(x)$ in any rank, either scalar, vector, tensor or $\cdots$, where $\partial_\lambda\,\psi(x)$ is the first partial derivative of the fields and $\partial_\lambda\, g_{\mu\nu}(x)$ is the first partial derivative of the metric tensor $g_{\mu\nu}$, the matter Lagrangian density is then given as [35]

$$\mathcal{L}_{\text{matter}}(\psi(x), \partial_\lambda\psi(x), \cdots; g_{\mu\nu}(x), \partial_\lambda g_{\mu\nu}(x), \cdots). \quad (21)$$

It is apparent that a quantum-induced correction of $\mathcal{L}_{\text{matter}}$ can be straightforwardly obtained when replacing $g_{\mu\nu}$ by $\tilde{g}_{\mu\nu}$, Eq. (14), which in turn is based on the minimal length uncertainty, and thereby on the possible quantum-induced corrections of the spacetime geometrical objects near the fundamental scales,

$$\tilde{\mathcal{L}}_{\text{matter}}(\psi(x), \partial_\lambda \psi(x), \cdots; \tilde{g}_{\mu\nu}(x), \partial_\lambda \tilde{g}_{\mu\nu}(x), \cdots). \tag{22}$$

In this regard, we focus on two physical examples:

- Electromagnetic field in curved spacetime [36]

$$\mathcal{L}_{\text{EM}} = -\frac{1}{4} g^{\mu\nu} g^{\alpha\beta} F_{\mu\nu} F_{\alpha\beta}, \tag{23}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the Faraday tensor, with ∂_μ is four-gradient and A_μ is four-potential. For quantum-induced metric tensor, Eq. (14), Eq. (23) can be rewritten as

$$\begin{aligned} \tilde{\mathcal{L}}_{\text{EM}} &= -\frac{1}{4} \tilde{g}^{\mu\nu} \tilde{g}^{\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \\ &= -\frac{1}{4} (1 + \mathcal{T}|\ddot{x}|^2)^{-2} g^{\mu\nu} g^{\alpha\beta} F_{\mu\nu} F_{\alpha\beta} \\ &= (1 + \mathcal{T}|\ddot{x}|^2)^{-2} \mathcal{L}_{\text{EM}}. \end{aligned} \tag{24}$$

Also here, for vanishing β_0 and/or $|\ddot{x}|^2$, the classical $\mathcal{L}_{\text{EM}}$ can be fully retrieved.

- Klein Gordon field (scalar field) in curved spacetime [37],

$$\mathcal{L}_\phi = -\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi), \tag{25}$$

where ϕ is the scalar field and $V(\phi)$ is the potential field. When replacing $g_{\mu\nu}$ by $\tilde{g}_{\mu\nu}$, Eq. (14), we get the quantum-induced version of the Klein-Gordon Lagrangian density

$$\begin{aligned} \tilde{\mathcal{L}}_\phi &= -\frac{1}{2} \tilde{g}^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \\ &= -\frac{1}{2} (1 + \mathcal{T}|\ddot{x}|^2)^{-1} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi). \end{aligned} \tag{26}$$

By substituting $-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi$ from Eq. (25), the quantum-induced scalar Lagrangian density reads

$$\tilde{\mathcal{L}}_\phi = \frac{1}{1+\mathcal{T}|\ddot{x}|^2} \left[\mathcal{L}_\phi - \mathcal{T}|\ddot{x}|^2 V(\phi)\right]. \tag{27}$$

For vanishing β_0 and/or $|\ddot{x}|^2$, the classical scalar Lagrangian density L_ϕ is straightforwardly obtained.

<u>D. Quantum-induced stress-energy tensor with EM Lagrangian density</u>

For now, Eq. (24) could be substituted into Eq. (19). Then, the quantum-induced stress-energy tensor with EM Lagrangian density reads

$$\begin{aligned} \tilde{T}_{\mu\nu} &= \frac{-2}{1 - \mathcal{T} g^{\mu\nu} \left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)} \frac{\partial \mathcal{L}_{\text{EM}}}{\partial g^{\mu\nu}} \\ &+ \frac{4\mathcal{T}\left(\ddot{x}_\mu \ddot{x}_\nu + \ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)}{(1+\mathcal{T}|\ddot{x}|^2)\left[1 - \mathcal{T} g^{\mu\nu}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]} \mathcal{L}_{\text{EM}} \\ &+ \frac{g_{\mu\nu}}{(1+\mathcal{T}|\ddot{x}|^2)} \mathcal{L}_{\text{EM}}. \end{aligned} \tag{28}$$

By substituting Eq. (16) into Eq. (28), we get

$$\tilde{T}_{\mu\nu} = \frac{1}{1 - \mathcal{T} g^{\mu\nu} \left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)} \left(T_{\mu\nu} - g_{\mu\nu} \mathcal{L}_{\text{EM}}\right)$$

$$+\frac{4\mathcal{T}\left(\ddot{x}_\mu\ddot{x}_\nu+\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\mu\nu}}+\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\mu\nu}}\right)}{(1+\mathcal{T}|\ddot{x}|^2)\left[1-\mathcal{T}g^{\mu\nu}\left(\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\mu\nu}}+\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]}\mathcal{L}_{\rm EM}$$

$$+\frac{g_{\mu\nu}}{(1+\mathcal{T}|\ddot{x}|^2)}\mathcal{L}_{\rm EM}. \qquad (29)$$

For $\partial\mathcal{L}_{\rm EM}/\partial g^{\mu\nu}=-\frac{1}{2}\left(T_{\mu\nu}-g_{\mu\nu}\mathcal{L}_{\rm EM}\right)$, Eq. (29) can be rewritten as

$$\tilde{T}_{\mu\nu}=\left[1-\mathcal{T}g^{\mu\nu}\left(\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\mu\nu}}+\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]^{-1}T_{\mu\nu}$$

$$-\left[1-\mathcal{T}g^{\mu\nu}\left(\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\mu\nu}}+\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]^{-1}g_{\mu\nu}\,\mathcal{L}_{\rm EM}$$

$$+\frac{4\mathcal{T}\left(\ddot{x}_\mu\ddot{x}_\nu+\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\mu\nu}}+\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\mu\nu}}\right)}{(1+\mathcal{T}|\ddot{x}|^2)\left[1-\mathcal{T}g^{\mu\nu}\left(\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\mu\nu}}+\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]}\mathcal{L}_{\rm EM}$$

$$+\frac{g_{\mu\nu}}{(1+\mathcal{T}|\ddot{x}|^2)}\mathcal{L}_{\rm EM}. \qquad (30)$$

It is noteworthy to highlight that the first line of Eq. (30) gives the classical stress-energy tensor $T_{\mu\nu}$ multiplied by a coefficient depending on quantum mechanical quantities including β_0, gravitational quantities including the first-order derivatives of the tangent covectors $\ddot{x}$, and the derivatives of $\ddot{x}$ with respect to $g_{\mu\nu}$. The second, third, and fourth lines of Eq. (30) refer to the electromagnetic Lagrangian density in curved spacetime multiplied by different coefficients depending on $\ddot{x}$ and its derivatives with respect to $g_{\mu\nu}$. Last but not least, at vanishing β_0 and/or $|\ddot{x}|^2$, the entire $\mathcal{L}_{\rm EM}$-contributions outlined in the second, third, and fourth lines of Eq. (30), vanish due to their vanishing coefficients. Also, at vanishing β_0 and/or $|\ddot{x}|^2$, the coefficient of $T_{\mu\nu}$ in the first line of of Eq. (30) becomes unity, so that $\tilde{T}_{\mu\nu}=T_{\mu\nu}$ is fully regained.

Furthermore by substituting Eq. (24) into Eq. (30), $\tilde{T}_{\mu\nu}$ can be simplified to

$$\tilde{T}_{\mu\nu}=\left[1-\mathcal{T}g^{\mu\nu}\left(\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\mu\nu}}+\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]^{-1}$$

$$\left\{T_{\mu\nu}-(1+\mathcal{T}|\ddot{x}|^2)\mathcal{T}\left[|\ddot{x}|^2\,g_{\mu\nu}-4\frac{|\ddot{x}|^2}{g_{\mu\nu}}-3\left(\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\mu\nu}}+\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\mu\nu}}\right)\right]\mathcal{L}_{\rm EM}\right\}, \qquad (31)$$

from which we conclude that the quantum-induced version of the stress-energy tensor with EM Lagrangian density is achieved through

- a linear factorization to $T_{\mu\nu}$ itself and
- a simultaneous emergence of $\mathcal{L}_{\rm EM}$ contributions.

The latter are also linearly factorized with quantities depending on β_0, $|\ddot{x}|^2$, $\mathcal{T}|\ddot{x}|^2$, $g_{\mu\nu}$, and variations of the first-order derivatives of tangent covectors with respect to $g^{\mu\nu}$. Whether the $\mathcal{L}_{\rm EM}$ contributions are subtracted or added to $T_{\mu\nu}$ depends on the coefficients inside the squared brackets in front of $\mathcal{L}_{\rm EM}$. To remain within the scope of the present paper, the nature and significance of all these quantum-induced corrections could be studied elsewhere.

1. Symmetry property of quantum-induced stress-energy tensor with EM Lagrangian

Given the symmetry of $\mathcal{L}_{\rm EM}$ and by exchanging the covariant indices μ and ν of Eq. (31), we get

$$\tilde{T}_{\nu\mu}=\left[1-\mathcal{T}g^{\nu\mu}\left(\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\nu\mu}}+\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\nu\mu}}\right)\right]^{-1}$$

$$\left\{T_{\nu\mu}-(1+\mathcal{T}|\ddot{x}|^2)\left[g_{\nu\mu}\mathcal{T}|\ddot{x}|^2-4\mathcal{T}|\ddot{x}|^2\frac{1}{g_{\nu\mu}}-3\mathcal{T}\left(\ddot{x}^\nu\frac{\partial\ddot{x}_\nu}{\partial g^{\nu\mu}}+\ddot{x}^\mu\frac{\partial\ddot{x}_\mu}{\partial g^{\nu\mu}}\right)\right]\mathcal{L}_{\rm EM}\right\}. \qquad (32)$$

Similarly, from the symmetry of the metric tensor $g_{\mu\nu}=g_{\nu\mu}$ and its inverse $g^{\mu\nu}=g^{\nu\mu}$,

$$\tilde{T}_{\nu\mu} = \left[1 - \mathcal{T} g^{\mu\nu}\left(\ddot{x}^{\mu}\frac{\partial \ddot{x}_{\mu}}{\partial g^{\mu\nu}} + \ddot{x}^{\nu}\frac{\partial \ddot{x}_{\nu}}{\partial g^{\mu\nu}}\right)\right]^{-1}$$

$$\left\{T_{\mu\nu} - (1+\mathcal{T}|\ddot{x}|^2)\left[g_{\mu\nu}\mathcal{T}|\ddot{x}|^2 - \quad 4\mathcal{T}|\ddot{x}|^2\frac{1}{g_{\mu\nu}} - 3\mathcal{T}\left(\ddot{x}^{\mu}\frac{\partial \ddot{x}_{\mu}}{\partial g^{\mu\nu}} + \ddot{x}^{\nu}\frac{\partial \ddot{x}_{\nu}}{\partial g^{\mu\nu}}\right)\right]\mathcal{L}_{\mathrm{EM}}\right\}$$

$$= \tilde{T}_{\mu\nu}. \qquad (33)$$

Thus, we conclude that similar to the classical stress-energy tensor [38], the quantum-induced version of the stress-energy tensor outlined in Eq. (31) is also symmetric under the exchange of the lower indices.

2. Covariant derivative of quantum-induced stress-energy tensor with EM Lagrangian

Because the covariant derivative of a tensor in one frame is the same in all other frames, for the sack of consistency, we need to investigate whether this applies to quantum-induced stress-energy tensors. For this purpose and for the sake of simplicity, we derive the covariant derivative of the stress-energy tensor with electromagnetic Lagrangian density in the free-falling frame. As the Faraday tensor is antisymmetric, i.e., $F_{\mu\nu} = -F_{\nu\mu}$, the covariant and partial derivatives are the same. Thus, Eq. (30) can be expressed as

$$\tilde{T}_{\mu\nu} = T_{\mu\nu} - \eta_{\mu\nu}\mathcal{L}_{EM} + \frac{4\mathcal{T}\ddot{x}_{\mu}\ddot{x}_{\nu} + \eta_{\mu\nu}}{(1+\mathcal{T}|\ddot{x}|^2)}\mathcal{L}_{EM}. \qquad (34)$$

With $\frac{\partial \ddot{x}_{\mu}}{\partial \eta^{\mu\nu}} = \frac{\partial \ddot{x}_{\nu}}{\partial \eta^{\mu\nu}} = 0$ and by replacing $g_{\mu\nu}$ with $\eta_{\mu\nu}$, then the covariant derivative reads

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = -\nabla^{\mu}(\eta_{\mu\nu}\mathcal{L}_{EM}) + \nabla^{\mu}\left[\frac{4\mathcal{T}\ddot{x}_{\mu}\ddot{x}_{\nu} + \eta_{\mu\nu}}{(1+\mathcal{T}|\ddot{x}|^2)}\mathcal{L}_{EM}\right], \qquad (35)$$

because $\nabla^{\mu}T_{\mu\nu}$ locally vanishes. Eq. (35) leads to

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = -\eta_{\mu\nu}\nabla^{\mu}\mathcal{L}_{EM} + \left[\frac{4\mathcal{T}\ddot{x}_{\mu}\ddot{x}_{\nu} + \eta_{\mu\nu}}{(1+\mathcal{T}|\ddot{x}|^2)}\right]\nabla^{\mu}\mathcal{L}_{EM} + \left[\frac{(1+\mathcal{T}|\ddot{x}|^2)(4\mathcal{T}\ddot{x}_{\nu}\nabla^{\mu}\ddot{x}_{\mu} + 4\mathcal{T}\ddot{x}_{\mu}\nabla^{\mu}\ddot{x}_{\nu})}{(1+\mathcal{T}|\ddot{x}|^2)^2}\right]\mathcal{L}_{EM}, \qquad (36)$$

where $(|\ddot{x}|^2)_{,\mu} = \eta_{\mu\nu,\mu}\ddot{x}^{\mu}\ddot{x}^{\nu} = 0$, $\nabla^{\mu}\eta_{\mu\nu} = \eta_{\mu\nu,\mu} = 0$, $\nabla^{\mu}\ddot{x}_{\mu} = \ddot{x}_{\mu,\mu} = 0$, and $\nabla^{\mu}\ddot{x}_{\nu}$. Then,

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = \left\{\left[\frac{4\mathcal{T}|\ddot{x}|^2 + \eta_{\mu\nu}}{(1+\mathcal{T}|\ddot{x}|^2)}\right] - \eta_{\mu\nu}\right\}\nabla^{\mu}\mathcal{L}_{EM}. \qquad (37)$$

So far, we conclude that $\nabla^{\mu}\tilde{T}_{\mu\nu}$ results in the covariant derivative of the Lagrangian density multiplied by coefficients depending on the metric tensor, β_0, $|\ddot{x}|^2$, and $\mathcal{T}$. If the quantization is removed by assigning zero to β_0 and/or $|\ddot{x}|^2$, $\nabla^{\mu}\tilde{T}_{\mu\nu}$ vanishes as well, even at finite $\nabla^{\mu}\mathcal{L}_{EM}$. The latter is defined in Eq. (23) in the free falling frame, for which g is to be replaced by η,

$$\nabla^{\mu}\mathcal{L}_{EM} = -\frac{1}{4}\eta^{\mu\nu}\eta^{\alpha\beta}\left(F_{\alpha\beta}F_{\mu\nu,\mu} + F_{\mu\nu}F_{\alpha\beta,\mu}\right). \qquad (38)$$

Maxwell's theory of electromagnetism implies that $\partial_{\mu}F^{\mu\nu} = \mu_0 J^{\nu}$ with μ_0 is the permeability of free space and J^{ν} is four-current. From Eq. (37) and Eq. (38), we get

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = \left\{\left[\frac{4\mathcal{T}|\ddot{x}|^2\eta_{\mu\nu} + \eta_{\mu\nu}}{(1+\mathcal{T}|\ddot{x}|^2)}\right] - \quad \eta_{\mu\nu}\right\}\left[-\frac{1}{4}\eta^{\mu\nu}\eta^{\alpha\beta}\left(F_{\alpha\beta}F_{\mu\nu,\mu} + F_{\mu\nu}F_{\alpha\beta,\mu}\right)\right], \qquad (39)$$

where $F_{\mu\nu,\mu} = \left(\eta_{\mu\alpha}F^{\alpha\beta}\eta_{\nu\beta}\right)_{,\mu}$ and $F_{\alpha\beta,\mu} = \left(\eta_{\alpha\mu}F^{\mu\nu}\eta_{\beta\nu}\right)_{,\mu}$. So far, we have concluded that

- for classical stress-energy tensor, i.e., vanishing β_0 and/or $|\ddot{x}|^2$, then

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = 0, \qquad (40)$$

- for quantum-induced stress-energy tensor, i.e., finite β_0 and/or $|\ddot{x}|^2$, in vacuum spacetime, i.e., vanishing four-current, then

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = 0, \qquad (41)$$

as $F_{\mu\nu,\mu} = F_{\alpha\beta,\mu} = 0$. Otherwise, $\nabla^\mu \tilde{T}_{\mu\nu}$ is divergent! In that case, the Lagrangian density should be coupled to a charged particle as a source in the inhomogeneous Maxwell equations so that

$$\nabla^\mu (\tilde{T}_{\mu\nu}^{\text{EM}} + \tilde{T}_{\mu\nu}^{\text{particle}}) = 0. \qquad (42)$$

These results are logically consistent with the Noether theorem for translation [39] in that every symmetry in Nature implies a conservation law. The vanishing covariant derivative of the stress-energy tensor originated in special relativity, i.e., $\delta^\mu T_{\mu\nu} = 0$. Noether theorem is not the main reason for carrying this result over GR. This is straightforwardly based on the fact that GR and special relativity are locally the same, so in local frames, e.g., free-falling frame, $\delta^\mu T_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0$.

For now, we can also interpret vanishing $\nabla^\mu \tilde{T}_{\mu\nu}$ to be due to diffeomorphism invariance. As long as the metric tensor could not be associated with a Killing vector field, no conserved quantity could be related to $\nabla^\mu \tilde{T}_{\mu\nu} = \nabla^\mu T_{\mu\nu} = 0$ [38].

E. Quantum-induced stress-energy tensor with scalar Lagrangian density

For spacetime filled with scalar field and by substituting Eq. (26) into Eq. (19), the quantum-induced stress-energy tensor can be expressed as

$$\tilde{T}_{\mu\nu} = \frac{-2(1+\mathcal{T}|\ddot{x}|^2)^2}{1-\mathcal{T} g^{\mu\nu}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)} \frac{\partial}{\partial g^{\mu\nu}}\left[-\frac{1}{2}(1+\mathcal{T}|\ddot{x}|^2)^{-1} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right]$$
$$+(1+\mathcal{T}|\ddot{x}|^2) g_{\mu\nu}\left[-\frac{1}{2}(1+ \qquad \mathcal{T}|\ddot{x}|^2)^{-1} g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - V(\phi)\right],$$

(43)

where $\ddot{x}^\mu = g^{\mu\nu}\ddot{x}_\nu$ and $\ddot{x}^\nu = g^{\mu\nu}\ddot{x}_\mu$.

$$\tilde{T}_{\mu\nu} = -\frac{2(1+\mathcal{T}|\ddot{x}|^2)^2}{1-\mathcal{T} g^{\mu\nu}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)} \frac{\partial L_\phi}{\partial g^{\mu\nu}} + g_{\mu\nu} L_\phi - \frac{1}{2}\frac{\mathcal{T}(1+\mathcal{T}|\ddot{x}|^2)^{-2}|\ddot{x}|^2 + \mathcal{T}(1+\mathcal{T}|\ddot{x}|^2)^{-2}}{1-\mathcal{T} g^{\mu\nu}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right) g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathcal{T}|\ddot{x}|^2\, V(\phi). \qquad (44)$$

For vanishing β_0 and/or $|\ddot{x}|^2$, the right-hand-side of Eq. (44) turns to express classical $T_{\mu\nu}$, where the first line gets a unity factor, i.e., the entire coefficient $(1+\mathcal{T}|\ddot{x}|^2)[1-\mathcal{T} g^{\mu\nu}(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}})]^{-1} \to 1$. Also, both the second and third lines entirely vanish. The nature and significance of the additional contributions associated with the quantum corrections, namely the second and third lines, shall be studied elsewhere.

1. Symmetry property of quantum-induced stress-energy tensor with scalar Lagrangian density

By exchanging the indices μ and ν, Eq. (44) reads

$$\tilde{T}_{\nu\mu} = -\frac{2(1+\mathcal{T}|\ddot{x}|^2)^2}{1-\mathcal{T} g^{\nu\mu}\left(\ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\nu\mu}} + \ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\nu\mu}}\right)} \frac{\partial L_\phi}{\partial g^{\nu\mu}} + g_{\nu\mu} L_\phi - \frac{1}{2}\frac{\mathcal{T}(1+\mathcal{T}|\ddot{x}|^2)^{-2}|\ddot{x}|^2 + \mathcal{T}|\ddot{x}|^2(1+\mathcal{T}|\ddot{x}|^2)^{-2}}{1-\mathcal{T} g^{\nu\mu}\left(\ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\nu\mu}} + \ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\nu\mu}}\right)}\left(\ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\nu\mu}} + \ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\nu\mu}}\right) g^{\nu\mu}\nabla_\nu\phi\nabla_\mu\phi - g_{\nu\mu}\mathcal{T}|\ddot{x}|^2\, V(\phi). \qquad (45)$$

The symmetry property of $g_{\mu\nu} = g_{\nu\mu}$ and $g^{\mu\nu} = g^{\nu\mu}$ leads to

$$\tilde{T}_{\nu\mu} = -\frac{2(1+\mathcal{T}|\ddot{x}|^2)^2}{1-\mathcal{T} g^{\mu\nu}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)} \frac{\partial L_\phi}{\partial g^{\mu\nu}} - \frac{1}{2}\frac{\mathcal{T}(1+\mathcal{T}|\ddot{x}|^2)^{-2}|\ddot{x}|^2 + \mathcal{T}|\ddot{x}|^2(1+\mathcal{T}|\ddot{x}|^2)^{-2}}{1-\mathcal{T} g^{\mu\nu}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right)}\left(\ddot{x}^\mu \frac{\partial \ddot{x}_\mu}{\partial g^{\mu\nu}} + \ddot{x}^\nu \frac{\partial \ddot{x}_\nu}{\partial g^{\mu\nu}}\right) g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - g_{\mu\nu}\mathcal{T}|\ddot{x}|^2\, V(\phi) + g_{\mu\nu} L_\phi. \qquad (46)$$

From Eqs. (45) and (46) we conclude that $\tilde{T}_{\mu\nu}$ with scalar Lagrangian fulfills the symmetry property.

2 *Covariant derivative of quantum-induced stress-energy tensor with scalar Lagrangian*

Again we recall that Noether theorem dictates that the differentiable symmetry is related to conservation laws in the underlying theory [39]. The differentiable symmetry is that of the action of a physical system defining the behavior of that system by the principle of least action. In section 2.5.1, we have shown that $\tilde{T}_{\mu\nu} = \tilde{T}_{\nu\mu}$. Thus, we categorically conclude that $\tilde{T}_{\mu\nu}$ fulfills the conservation laws, locally.

On the one hand, the angular momentum conservation is satisfied by the symmetry property of spacetime indices of the quantum-induced stress-energy tensor [40]. On the other hand, the energy-momentum conservation is fulfilled by vanishing covariant derivative of the stress-energy tensor with respect to the spacetime. The quantum-induced version of the stress-energy tensor introduced in the present study is defined from the Hilbert action, $\delta S = 0$. Vanishing variation of S refers to the diffeomorphism invariance property, i.e., the stress-energy tensor is also diffeomorphism invariant, and its covariant derivative leads to

$$\nabla^{\mu}\tilde{T}_{\mu\nu} = 0. \qquad (47)$$

i.e., the stress-energy tensor is not explicitly dependent on the coordinates x^{μ}, i.e, its divergence is zero; it is locally conserved.

<u>III. Summary and Conclusion</u>

Several quantum gravity approaches lead to a minimal length scale scenario that is conjectured to integrate gravity in quantum mechanics through a generalized uncertainty principle (GUP), which is a generalization of the Heisenberg uncertainty principle and can be used to help incorporate the quantum effects of the gravitational interaction on the spacetime geometry near the minimal scale of the length at which such effects are expected to become important. The principal geometric objects of the classical representation of spacetime geometry in the classical theory of general relativity (GR) are eventually likely to become deformed or modified.

In the present paper, the minimal measurable length scale L is taken to be the minimal uncertainty of distance Δx_0 in Kempf's GUP model [26], which enables the quantum-induced deformation of geometrical structures on spacetime manifold M. In a recent paper [25], we used the recipe that was first proposed by Caianiello [41] and later developed by Brandt [24] to calculate the quantum-induced fundamental (metric) tensor. In a more recent work [22], we pursued the same reasoning and calculated the quantum-corrected Levi-Civita connection and the Reimann curvature tensor.

Unlike the approaches by Caianiello [23] and Brandt [24], our method is independent of any particle mass scale. It expresses a minimal length scale, which can be naturally related to a maximal acceleration scale by some quantum deformation parameter and/or basic physical constants, depending on which minimal length scale is chosen. Additionally, our method relies on observers for the metric and other geometric objects. It is therefore not essential to employ a universal length scale or mass scale to define the minimum length (or maximum acceleration). In other words, our approach assumes that barring phenomenological constraints, we do not need to identify the minimum length scale to be precisely Planck scale.

Through the minimal length uncertainty and the additional curvature imposed on the 8-dimensional tangent bundle (phase-space), the quantum effects of the gravitational fields can be embedded in the four-dimensional classical pseudo-Riemann manifold of GR. The most general geometric length measure for curves is thereby assured, especially when relaxing invariance under local Lorentz transformation. The generalized (quantum-induced) fundamental (metric)

tensor is determined, $(\sqrt{-|g|}\hbar\sqrt{\beta_0})\dot{x}^{\mu})$ on the tangent bundle. To summarize, GR is assumed to be quantum-corrected near some minimum (fundamental) length scale. Therefore, the main geometrical elements of the classical theory of general relativity, including the stress-energy tensor, are eventually anticipated to become deformed or corrected. The quantum-induced version of the stress-energy tensor is constructed through a complementary term reconciling the principles of QM and GR and comprising generalized noncommutative Heisenberg algebra.

As we showed in the first line of Eq. (30), the quantum-induced stress-energy tensor is given by the classical stress-energy tensor $T_{\mu\nu}$ multiplied by a coefficient depending on quantum mechanical and gravitational quantities including the derivative of the tangent convectors $\ddot{x}$ [1] and the derivatives of $\ddot{x}$ with respect to $g_{\mu\nu}$. The second, third, and fourth lines of Eq. (30) refer to the electromagnetic Lagrangian density in curved spacetime multiplied by different coefficients depending on $\ddot{x}$ and its derivatives with respect to $g_{\mu\nu}$. When β_0 and/or $|\ddot{x}|^2$ vanish, the entire $\mathcal{L}_{\rm EM}$-contributions outlined in the second, third, and fourth lines of Eq. (30) vanish. Also, at vanishing β_0 and/or $|\ddot{x}|^2$, the coefficient of $T_{\mu\nu}$ in the first line becomes unity, so that $\tilde{T}_{\mu\nu} = T_{\mu\nu}$ is fully restored. We also found that the quantum-induced version of the stress-energy tensor with EM lagrangian outlined in Eq. (31), like its classical counterpart, is also symmetric under the change of the two lower indices. The nature and significance of all these quantum-induced corrections could be studied elsewhere.

We constructed the stress-energy tensor for Lagrangian density with electromagnetic and scalar fields. For both fields, the 'quantization' is realized through i) a linear factorization to the stress-energy tensor itself, and ii) simultaneous emergence of additional contributions of the Lagrangian densities. The latter are also linearly factorized with quantities depending on GUP parameter β_0, the second-order derivatives of tangent covectors $|\ddot{x}|^2$, the fundamental tensor $g_{\mu\nu}$, and the derivatives of $|\ddot{x}|^2$ with respect to $g^{\mu\nu}$.

We have shown that both quantum-induced stress-energy tensors with electromagnetic and scalar Lagrangian densities are symmetric in their covariant indices. For the quantum-induced stress-energy tensor with electromagnetic Lagrangian density in vacuum spacetime, we have derived the covariant derivative and categorically concluded its vanishing divergence. Otherwise, the Lagrangian density should be coupled with the matter source in the inhomogeneous Maxwell's equation. For the quantum-induced stress-energy tensor with scalar Lagrangian density, the conclusion of the vanishing covariance derivative is based on the Noether theorem for translation that very symmetry implies conservation.

Fortunately, the 'quantization' proposed in the present paper is linearly factorized to the classical stress-energy tensor. The classical version of the stress-energy tensor, Einstein GR's version, is the basis and is always present. The quantum-corrected version appears as supplementary, which could be switched off. In both cases, the classical stress-energy tensor exists. On the other hand, the quantum correction comes up with coefficients, functions of quantum-mechanical and relativistic quantities, that differ from unity in front of the classical stress-energy tensor, and also additional contributions from the Lagrangian densities and potentials. The latter have coefficients of quantum-mechanical and relativistic quantities. The nature and significance of all these coefficients and the additional Lagrangian densities and potentials shall be studied elsewhere.

At the macroscopic level, where $\hbar \to 0$, the modification (deformation) of the EM Lagrangian, EM stress-energy tensor, scalar field Lagrangian, and scalar field stress-energy tensor, vanishes, and their classical GR forms are restored. This ensures that such modifications have a quantum origin that manifests exclusively at the appropriate fundamental scales where $L^2\ddot{x}^2$ becomes significant. The extra curvature of the deformed manifold exists if and only if $\ddot{x}^\mu \neq 0$, and it is entirely due to the proper acceleration of the test particle rather than any other matter or energy–momentum sources.

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Declaration of Conflict of Interest

The authors declare that there is no conflict of interest.

Ethical approval

This study follows the ethics guidelines of the Faculty of Science, Helwan University, Egypt (ethics approval number: REC-Sci-HU/P29-04-02.

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تنسور الإجهاد والطاقة الناتج عن التأثيرات الكمومية على نظام نسبوي

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الملخص

يفرض الجبر اللاتبادلي المعمم لهيزنبرغ، استنادًا إلى مبدأ عدم اليقين المعمم، حدًا أدنى لعدم اليقين على ميكانيكا الكم. من ناحية أخرى، يُقترح أن تعديل الزمكان الناتج عن التأثير الكمومي المعمم يمثل انحناءً إضافيًا في الأبعاد الثمانيه (أربعة أبعاد للزمكان الكلاسيكي و أربعة أبعاد لفضاء الطور). يظهر هذا التعديل على شكل حد جبرإضافي مكمل للحدود الكلاسيكية يجمع بين تاثيرات ميكانيكا الكم والنسبية العامة ويشمل الحد الأدنى للطول والاشتقاقات من الدرجة الأولي لمتجه السرعة. بناءً على ذلك، يمكن بناء تنسور القياس المعدل الخالي من الالتواء. كذلك، تم تعديل تنسور الإجهاد والطاقة المتماثل، والذي يُعتبر مصدر انحناء الزمكان، بالإضافة إلى كثافة الطاقة المرتبطة بلاجرانج المجال الكهرومغناطيسي و المجال القياسي. بجانب النسخة الكلاسيكية لتنسور الإجهاد والطاقة، يتم إدخال كثافات لاجرانج إضافية وطاقات جهد، مع معاملات تعتمد على تنسور القياس، واشتقاقات متجه السرعة، والثوابت الفيزيائية التي تشمل ثابت الجاذبية، وثابت بلانك، وسرعة الضوء، وطول بلانك. يأكد اشتقاق تنسور الإجهاد والطاقة المعدّل بالتأثير الكمومي من تحقق مبادي نظرية النسبية العامة لأينشتاين، ويقترح أن معادلة الاستمرارية تشير إلى أن حقول الجاذبية تبذل شغلا على المادة الكلاسيكية والكمومية والعكس بالعكس. عند تلاشي المشتقة الأولى لمتجه السرعة و/أو عدم وجود حد أدنى لعدم اليقين في الطول، يتم استعادة التعبيرات الرياضية المتعارف عليها في النسبية العامة الكلاسيكية وفي الميكانيكا الكمومية التقليدية بشكل كامل. بناءً على ذلك، يتم أيضًا استرجاع تنسور الإجهاد والطاقة لأينشتاين. لذلك، نستنتج أن تنسورالإجهاد والطاقة المعدّل المقترح مناسب من حيث المبدأ لكل من معادلات الحقول الكمومية والكلاسيكية.

الكلمات الدالة: الجاذبية المعدلة، مقياس الطول الأدنى، مبدأ عدم اليقين المعمم، النسبية العامة، تنسور الإجهاد والطاقة، الفضاء الزمكاني المعدل.