



Solving of Advection-Diffusion Equation in Three Dimensions in All Stabilities

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Abstract

In this work the Separation and Laplace transform methods are used to propose model solution for the equation of the Advection-diffusion in the three-dimensions in stable, neutral and unstable conditions (in all stabilities). Considering the mixing height is discretizing into N-sub-layers using the form of wind velocity and vertical turbulence in all stabilities. The wind speed, the lateral and the vertical turbulent diffusivities are considered the vertical height dependent. The inverse of Laplace transform is obtained by Gaussian Quadrature Scheme. The proposed concentrations were calculated using the proposed model in all stabilities. For unstable conditions the proposed concentrations were compared with the first 1st experimental data recorded for radioactive Iodine-135 (I^{135}) of the first reactor at Egyptian Atomic Energy Authority test at Inshas. While, for stable and neutral conditions the proposed concentrations were compared with the second 2nd experimental data of Iodine I-131 (I^{131}) released from the second research reactor. Taking into consideration that Comparing between the proposed model, previous work and observed concentrations of Iodine-135 in unstable, and Iodine-131 in neutral and stable conditions. The results show that there is a good perfect agreement between the proposed and observed isotope concentrations. Also, the statistical techniques show that the existence of a factor of two between the model proposed and the concentrations of the observed isotope. All results are represented by figures and tables.

Keywords: Laplace Transform; Stabilities Conditions; Separation Method; Gaussian Quadrature Scheme.

Introduction

Advection-diffusion equation in three-dimensional with Steady state was solved using Fourier transform considering vertical turbulent diffusivity was as function of linear downwind distance and constant wind speed to obtain a normalized crosswind integrated concentration [1]. Also, the above problem was obtained by assuming that the vertical turbulent diffusivity was as function of power law of vertical height [2]. Semi-analytical model for coupled multispecies adjective-dispersive transport subject to rate-limited had been studied [3]. The pollutant concentration was obtained using Hankel Transform in terms of a given flux of dust from the ground surface [4]. The analytical solution for the equation of advection-diffusion with variables for the diffusivity of vertical turbulent and wind speed through Hankel Transform was estimated [5]. Also, analytical treatment for the equation of the fractional advection diffusion in three dimensions was investigated [6].

In this work, the three-dimensional steady state advection-diffusion equation is solved using Laplace transform and separation of variables technique, taking the wind velocity, crosswind and vertical turbulent diffusivities as a function of vertical height in neutral, stable and unstable conditions respectively to get the concentrations. A comparison is obtained among the predicted concentrations, previous work and observed of radioactive Iodine I^{135} in unstable condition and I^{131} in neutral and stable conditions at Egyptian Atomic Energy Authority respectively. The predicted models are found to agree well the observed concentrations.

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Technique Method

The equation for the advection-diffusion in three-dimensions can be simplified as:

$$u \frac{\partial C}{\partial x} = \frac{\partial}{\partial y} \left(k_y \frac{\partial C}{\partial y} \right) + \frac{\partial}{\partial z} \left(k_z \frac{\partial C}{\partial z} \right) \quad (1)$$

where, the concentration in three dimensions is C (Bq/m³), k_y is the turbulent diffusivities in y direction, k_z is the turbulent diffusivities z directions and u depends on “z”.

Assuming the solution of Eq. (1) is in the following form:

$$C = Q C_y C_z \quad (2)$$

where, Q is the rate of emission (Bq). Substituting from Eq. (2), in Eq. (1), we can get the two equations:

$$u(z) \frac{\partial C_y(x, y)}{\partial x} = \frac{\partial}{\partial y} \left(k_y \frac{\partial C_y(x, y)}{\partial y} \right) \quad (3.1)$$

$$u(z) \frac{\partial C_z(x, y)}{\partial x} = \frac{\partial}{\partial z} \left(k_z \frac{\partial C_z(x, y)}{\partial z} \right) \quad (3.2)$$

The two Eqs. (3.1) and (3.2) are estimated under the following boundary conditions:

(a) There isn't vertical flux at surface and mixing height i.e.

$$k_z \frac{\partial C}{\partial z} = 0, \quad \text{at } z = 0, h \quad (4a)$$

(b) No flux in y-direction at $y=0$ and L_y i.e.

$$k_y \frac{\partial C}{\partial y} = 0, \quad \text{at } y = 0, L_y \quad (4b)$$

© There is mass continuity as follows:

$$u C_z = Q \delta(z - h_s) \quad \text{at } x = 0 \quad (4c1)$$

$$u C_y = Q \delta(y - y_o) \quad \text{at } x = 0 \quad (4c2)$$

where, y_o is a small distance in y-direction.

(d) There is no concentration at large distance as follows:

$$C \rightarrow 0 \quad \text{as } y \rightarrow \pm\infty \quad \text{and } z \rightarrow \infty \quad (4d)$$

where, h is the height of Planetary Boundary Layer (PBL) (m), L_y is a large distance in the y direction, and δ is a Dirac delta function.

Assuming that “h” is discretizing into N sub-intervals stepwise where, $k(z)$ and $u(z)$, taking the average values which are as follows:

$$k_n = \frac{1}{z_{n+1} - z_n} \int_{z_n}^{z_{n+1}} k_n(z) dz \quad (5)$$

$$u_n = \frac{1}{z_{n+1} - z_n} \int_{z_n}^{z_{n+1}} u(z) dz \quad (6)$$

Assume that the crosswind turbulence parameters is taken as the following form:

$$k_y(x, z) = \beta x u \quad (7)$$

First Equation Model

Now, the first Eq. (3.1) of this model is solved as the following:

Substituting from Eq. (7) in Eq. (3.1) one gets:

$$\frac{\partial C_y}{\partial x} = \beta x \frac{\partial^2 C_y}{\partial y^2} \quad (8)$$

Eq. (8) is calculated by the separation method, as follows:

$$C_y = \chi_l(x)\eta_l(y) \quad (9)$$

Then, Eq. (8), becomes:

$$\frac{1}{x} \frac{1}{\chi_l(x)} \frac{\partial \chi_l(x)}{\partial x} = \frac{\beta}{\eta_l(y)} \frac{\partial^2 \eta_l(y)}{\partial y^2} = -\lambda_l^2 \quad (10)$$

where, λ_l is a constant. Then, two differential equations are obtained as follows:

$$\frac{\partial \chi_l(x)}{\partial x} = -\lambda_l^2 x \chi_l(x) \quad (11a)$$

$$\frac{\partial^2 \eta_l(y)}{\partial y^2} = \frac{-\lambda_l^2}{\beta} \eta_l(y) \quad (11b) \text{ Eqs. (11)}$$

a) and (11 b) are evaluated as follows:

$$\chi_l(x) = a_1 e^{\frac{-\lambda_l^2 x^2}{2}} \quad (12a)$$

$$\eta_l(y) = a_2 \cos\left(\frac{\lambda_l}{\sqrt{\beta}} y\right) + a_3 \sin\left(\frac{\lambda_l}{\sqrt{\beta}} y\right) \quad (12b)$$

where, a_1, a_2 , and a_3 are constant values which estimated from the boundary condition (4b), then $a_3 = 0$ and $\lambda_l = \frac{l\pi\sqrt{\beta}}{L_y}$, $l = 0, 1, 2, \dots$

Then Eq. (8) has the following solution:

$$C_y = \sum_{l=0}^{\infty} B_l e^{\frac{-\lambda_l^2 x^2}{2}} \cos\left(\frac{l\pi}{L_y} y\right) \quad (13)$$

where, $B_l = a_1 a_2$, from Eqs. (4c) and (9), one gets: $B_0 = \frac{1}{L_y}$, $B_l = \frac{2}{L_y}$, $l = 1, 2, 3, \dots$

2.2 Second Equation Model

And then, the second Eq. (3.2) of this model is solved as the following:

From Eqs. (5) and (6), Eq. (3.2) will be:

$$\frac{k_n(z)}{u_n(z)} \frac{\partial^2 c_{zn}(x,z)}{\partial z^2} = \frac{\partial c_{zn}(x,z)}{\partial x}, \quad n = 1: N \quad (14)$$

Taking the Laplace transform on "x" with boundary conditions as follows:

$$C_{zn}(0, z_n) = \frac{Q}{u_n} \delta(z_n - h_s) \quad (i)$$

$$k_n(z) \frac{\partial c_{zn}(x,z)}{\partial z} = 0 \quad \text{at } z_n = 0, h \quad (ii)$$

Eq. (14) becomes:

$$\int_0^{\infty} u \frac{\partial c_{zn}}{\partial x} e^{-sx} dx = k_n(z) \int_0^{\infty} \frac{\partial^2 c_{zn}}{\partial z^2} e^{-sx} dx \quad (15)$$

Eq. (15), can be written as:

$$-u c_{zn}(0, z) + s u \tilde{c}_{zn}(s, z) = k_n(z) \frac{\partial^2 \tilde{c}_{zn}(s, z)}{\partial z^2} \quad (16)$$

Using the boundary condition (i), Eq. (16) becomes:

$$\frac{\partial^2 \tilde{c}_{zn}(s, z)}{\partial z^2} - \frac{s u}{k_n} \tilde{c}_{zn}(s, z) = -\frac{Q}{k_n} \delta(z_n - h_s) \quad (17)$$

Taking the Laplace transform on z then:

$$p^2 \tilde{\tilde{c}}_{zn}(s, p) - p c_{zn}(s, 0) - \frac{\partial \tilde{c}_{zn}(s, 0)}{\partial z} - \frac{us}{k_n} \tilde{\tilde{c}}_{zn}(s, p) = -\frac{Q}{k_n} e^{-ph_s} \quad (18)$$

Substituting the condition (ii), Eq. (18) becomes:

$$\tilde{c}_{zn}(s, p) = \frac{c_{zn}(s, 0)p}{(p^2 - \frac{us}{k_n})} - \frac{Q e^{-ph_s}}{k_n(p^2 - \frac{us}{k_n})} \quad (19)$$

$$\tilde{c}_{zn}(s, p) = c_{zn}(s, 0)F(s, p) - \frac{Q}{k_n} e^{-ph_s} G(s, p) \quad (20)$$

where, $F(s, p) = \frac{p}{(p^2 - \frac{us}{k_n})}$ and $G(s, p) = \frac{1}{(p^2 - \frac{us}{k_n})}$

Taking the inverse of Eq. (20) one gets:

$$\tilde{c}_{zn}(s, z) = \frac{c_{zn}(s, 0)}{2} \left[e^{\sqrt{\frac{su}{k_n}}z} + e^{-\sqrt{\frac{su}{k_n}}z} \right] - \frac{Q}{2k_n} \sqrt{\frac{k_n}{su}} \left[e^{\sqrt{\frac{su}{k_n}}(z-h_s)} - e^{-\sqrt{\frac{su}{k_n}}(z-h_s)} \right] H(z - h_s) \quad (21)$$

Let $R_n = \sqrt{\frac{su}{k_n}}$ and $R_a = \sqrt{su k_n}$

$$\tilde{c}_{zn}(s, z) = \frac{c_{zn}(s, 0)}{2} [e^{R_n z} + e^{-R_n z}] - \frac{Q}{2R_a} [e^{R_n(z-h_s)} - e^{-R_n(z-h_s)}] H(z - h_s) \quad (22)$$

$$\tilde{c}_{zn}(s, z) = c_{zn}(s, 0) \cosh R_n z - \frac{Q}{R_a} \sinh R_n(z - h_s) * H(z - h_s) \quad (23)$$

Using the boundary condition (ii) one gets:

$$\frac{\partial}{\partial z} \tilde{c}_{zn}(s, z) = R_n c_{zn}(s, 0) \sinh R_n z - \frac{Q}{R_a} R_n \cosh R_n(z - h_s) H(z - h_s) - \frac{Q}{R_a} \sinh R_n(z - h_s) \frac{\partial}{\partial z} H(z - h_s) \quad (24)$$

$$c_{zn}(s, 0) \sinh(R_n h) = \frac{Q}{R_a} \cosh(R_n(h - h_s)) H(h - h_s) \quad (25)$$

$$c_{zn}(s, 0) = \frac{Q \cosh R_n(h - h_s)}{R_a \sinh(R_n h)}$$

$$c_{zn}(s, 0) = \frac{Q}{\sqrt{su k_n}} \frac{\cosh \sqrt{\frac{su}{k_n}}(h - h_s)}{\sinh \sqrt{\frac{su}{k_n}} h} \quad (26)$$

Substituting from equation (26) in equation (23) then one gets:

$$\tilde{c}_{zn}(s, z) = \frac{Q}{\sqrt{su k_n}} \frac{\cosh \sqrt{\frac{su}{k_n}}(h - h_s)}{\sinh \sqrt{\frac{su}{k_n}} h} \cosh R_n z - \frac{Q}{R_a} \sinh R_n(z - h_s) * H(z - h_s) \quad (27)$$

The method of Gaussian quadrature formulas is used; one can get:

$$\frac{c_{zn}(x, z)}{Q} = \sum_{i=1}^8 a_i \left(\frac{p_i}{x} \right) \frac{1}{\sqrt{uk_n(z)p_i}} \frac{\cosh \sqrt{\frac{p_i u}{xk_n}}(z_i - h_s) \cosh(R_n z)}{\sinh \sqrt{\frac{p_i u}{xk_n}} z_i} \quad (28)$$

Using Eqs. (13) and (28), Eq. (1) is calculated as follows:

$$C(x, y, z) = \sum_{i=1}^8 a_i \left(\frac{p_i}{x} \right) \frac{Q}{\sqrt{uk_n(z)p_i}} \frac{\cosh \sqrt{\frac{p_i u}{xk_n}}(z_i - h_s) \cosh(R_n z)}{\sinh \sqrt{\frac{p_i u}{xk_n}} z_i} \sum_{l=0}^{\infty} B_l e^{-\frac{\lambda_l^2 x^2}{2} - \frac{ux}{u}} \cos \left(\frac{l\pi}{L_y} y \right) \quad (29)$$

where, $e^{-ux/u}$ is the radioactive decay for isotope.

Stable Condition

In the stable condition, the wind velocity and vertical turbulence parameters, take the forms as follows:

$$u(z) = \frac{u_*}{0.4} \left(\ln \frac{z}{z_0} - 5 \frac{z}{L} \right) \quad (30)$$

$$k_z(z) = 0.4 u_* \frac{z}{1+5\frac{z}{L}}, \quad (31)$$

where, $\beta = \left(\frac{0.4 W_*}{u} \right)^2$ is the turbulence constant, w_* is vertical velocity of convective and 'u' is a wind velocity (m/s).

Results one (stable)

The observed concentrations of I^{131} isotope was obtained from observed experiments of the Second Research Reactor at Egyptian Atomic Energy Authority at Inshas, in the stable condition. The observed samples were from a stack of height 27 m with a roughness length of 0.6 m. The meteorological data during the experiments are considered [7,8] which is given in Table (1). The observed concentrations and predicted concentrations below the plume centreline from Eqs. (7), (30) and (31) of I^{131} isotope are also given in Table (2) using Eq. (29). Figures (1) and (2) show that the proposed concentrations are in a good agreement with observed concentrations, and the existence of a factor of two between the proposed concentrations data and the observed concentrations data.

Table 1. The meteorological in the neutral and stable conditions at the second experiment (Essa, 2009).

Exp	Atmospheric stability	L	u_* (m/s)	u_{27} (m/s)	mixing height (h) (m)
1	D	∞	0.67	5.80	2680
5	E	55	0.50	3.80	209

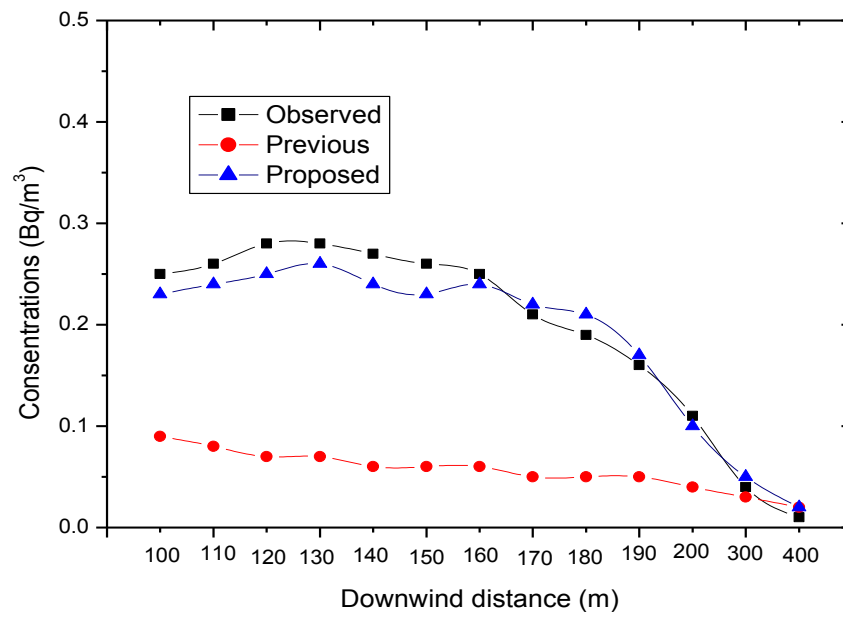


Fig. 1. The concentrations of proposed and observed of (I^{131}) via downwind distance in stable condition.

Table 2. Measured and proposed concentrations of I^{131} in the stable condition

Distance (m)	Observed (Bq/m ³)	Proposed Model	Previous work (Bq/m ³)
		Eq.(29) (Bq/m ³)	Essa et al. (2016)
100	0.25	0.23	0.09
110	0.26	0.24	0.08
120	0.28	0.25	0.07
130	0.28	0.26	0.07
140	0.27	0.24	0.06
150	0.26	0.23	0.06
160	0.25	0.24	0.06
170	0.21	0.22	0.05
180	0.19	0.21	0.05
190	0.16	0.17	0.05
200	0.11	0.10	0.04
300	0.04	0.05	0.03
400	0.01	0.02	0.02

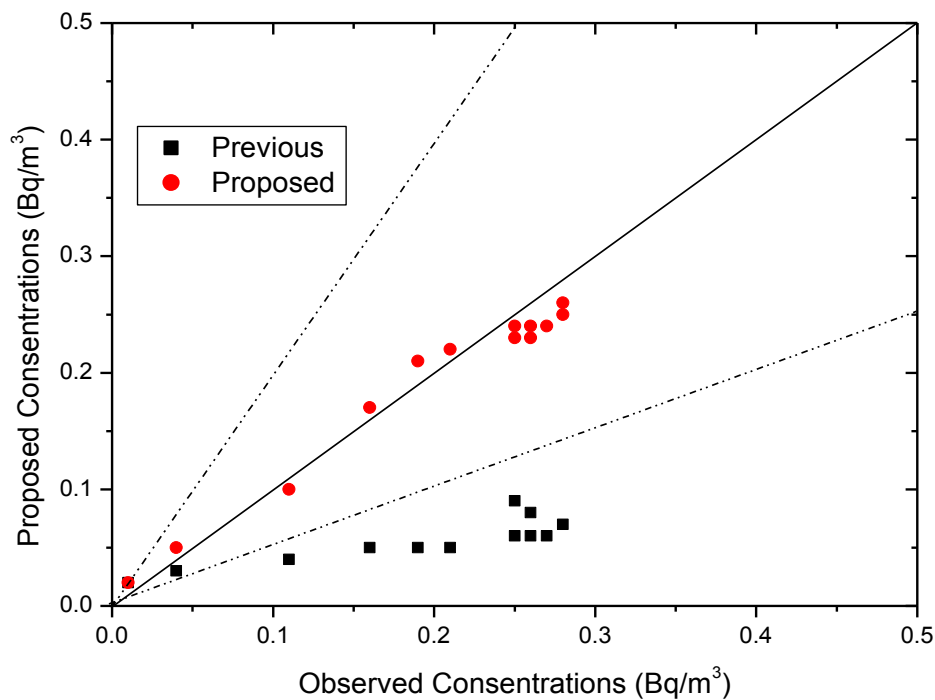


Fig. 2. The relation between the proposed and observed consternations of Iodine-131 (I^{131}) in the stable condition.

Statistical Analysis

Comparing between the proposed and observed concentrations was introduced in the statistical form [9], where, the Normalized Mean Square Error, the Fraction Bias, the Correlation Coefficient and the Factor of Two are denoted by NMSE, FB, COR and FAC2 respectively.

Table 3. Comparing between proposed and observed concentration in the stable condition.

Stable case	NMSE	FB	COR	FAC2
Proposed	0.08	0.09	0.99	0.96
Previous (stable)	2.29	1.12	0.81	0.08

It is shown that in Table (3), NMSE and FB close to zero, but COR and FAC2 close to one in stable condition. So that the proposed model inside a factor of two with observed concentration data and the proposed model achieved 96% from observed data.

Neutral Condition

In neutral case, the wind velocity and vertical turbulence parameters, take the forms as follows:

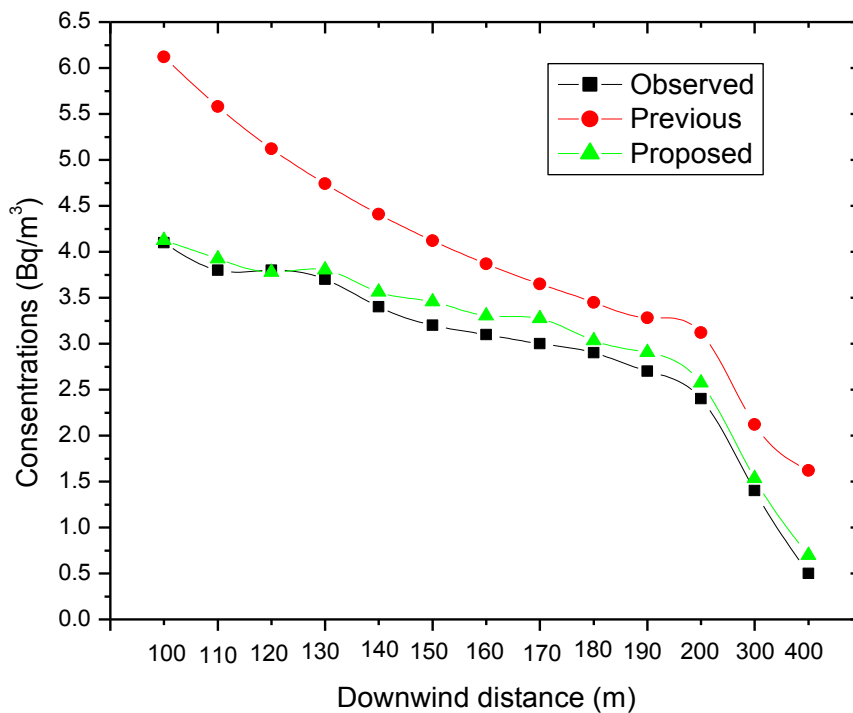
$$u_n = \frac{u_*}{0.4} \ln \left(\frac{z+z_0}{z_0} \right) \text{ and } k_n = 0.4 u_* z, \text{ then the meteorology in neutral and stable conditions are taken from Table (1).}$$

Results two (neutral)

The observed and proposed concentrations of Iodine-131 (I^{131}) in neutral case with downwind distance are presented in Table (4) as follows:

Table 4. Observed and proposed concentrations for I^{131} in neutral condition

Downwind distance (m)	Observed conc. (Bq/m^3)	Previous work conc. (Bq/m^3)	
		Essa et al. (2016)	Eq. (29) (Bq/m^3)
100	4.1	6.12	4.1259
110	3.8	5.58	3.92509
120	3.8	5.12	3.77909
130	3.7	4.74	3.80299
140	3.4	4.41	3.56152
150	3.2	4.12	3.45665
160	3.1	3.87	3.30269
170	3	3.65	3.27221
180	2.9	3.45	3.03374
190	2.7	3.28	2.90451
200	2.4	3.12	2.57199
300	1.4	2.12	1.53209
400	0.50	1.62	0.697924

**Fig. 3. The variation of I^{131} concentrations with downwind distance in neutral case.**

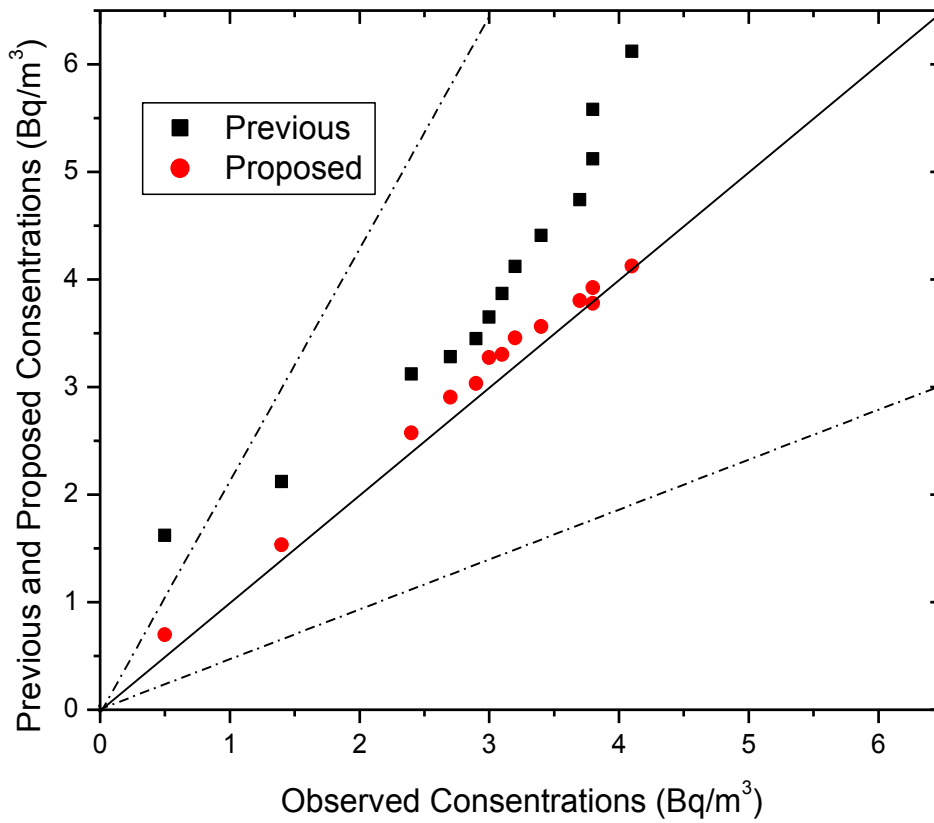


Fig. 4. The variability of previous and proposed concentrations with observed concentrations of I¹³¹ in neutral case.

Statistical Analysis

Table 5. The comparison between proposed and experimental concentrations in neutral condition.

Neutral case	NMSE	FB	COR	FAC2
Proposed	0.001	0.05	0.997	1.05
Previous (neutral)	0.11	-0.30	0.88	0.92

Unstable Condition

The wind velocity and the vertical turbulence are taken the forms in the unstable condition as follows:

$$u(z) = \frac{u_*}{0.4} \left(\ln \frac{z}{z_o} - 2 \ln \left[0.5 \left(1 + \frac{1}{\phi_m} \right) \right] - \ln \left[0.5 \left(1 + \frac{1}{\phi_m^2} \right) \right] + 2 \tan^{-1} \frac{1}{\phi_m} - \frac{\pi}{2} \right) \quad (32)$$

$$k_z(z) = 0.4 u_* \frac{z}{\phi_m \left(\frac{z}{L} \right)}, \quad (33)$$

where,
$$\phi_m \left(\frac{z}{L} \right) = \left(1 - 15 \frac{z}{L} \right)^{-1/4} \quad (34)$$

Substituting Eqs. (32), (33) and (34) in Eq. (29), the concentrations can be calculated in unstable condition as follows.

Results Three (unstable)

The air samples of observed isotope concentrations of I^{135} in unstable condition were collecting from the First Research Reactor at Inshas, Egyptian Atomic Energy Authority. The experiments were observed from a stack height is 43 m with a roughness length of 0.6 m. The meteorological data of I^{135} are considered [10]. The proposed concentrations by Eq. (29) below the plume centerline are also given in Table (6). One finds that the proposed model concentrations and observed concentrations are well agreement. As shown in Figs (5) and (6) there is a factor of two between the proposed concentrations data and the observed concentrations data in unstable condition.

Table 6. Meteorological data of the nine convective test runs in March and May 2006.

Run no.	Working hours of the source	Release rate (Bq)	Wind speed ($m s^{-1}$)	Wind direction(deg)	W_* (ms^{-1})	P-G stability class
1	48	1028571	4	301.1	2.27	A
2	49	1050000	4	278.7	3.05	A
3	1.5	42857.14	6	190.2	1.61	B
4	22	471428.6	4	197.9	1.23	C
5	23	492857.1	4	181.5	0.958	A
6	24	514285.7	4	347.3	1.3	D
7	28	1007143	4	330.8	1.51	C
8	48.7	1043571	4	187.6	1.64	C
9	48.25	1033929	4	141.7	2.1	A

Table 7. Observed and proposed concentrations of nine experiments

Test	Downwind distance (m)	Observed conc. (Bq/m^3)	Proposed model conc. Eq. (29) (Bq/m^3)	Previous work conc. (Bq/m^3) (Essa et al. 2008)
1	100	0.025	0.021	0.0296
2	98	0.037	0.031	0.0197
3	136	0.091	0.075	0.0508
4	135	0.197	0.187	0.2247
5	106	0.272	0.254	0.3339
6	186	0.188	0.165	0.1218
7	165	0.447	0.431	0.4159
8	154	0.123	0.154	0.1500
9	106	0.032	0.031	0.0381

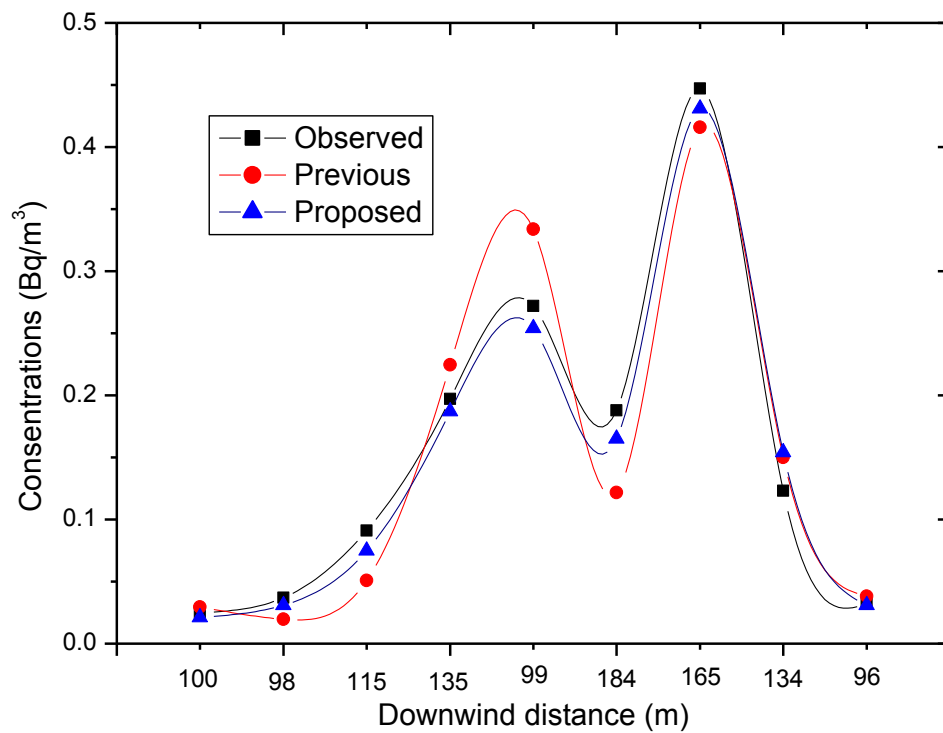


Fig. 5. The proposed and observed isotope concentrations of (I^{135}) via downwind distance in unstable condition.

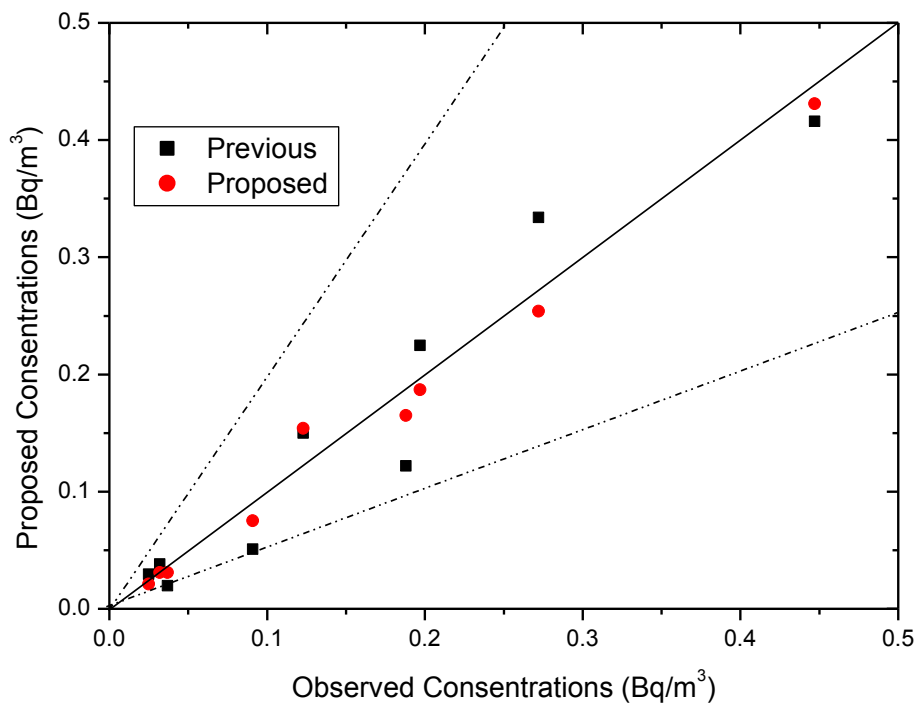


Fig. 6. The relation between the proposed and observed consternations of (I^{135}) in unstable condition.

Statistical Analysis

Table 8. The comparison between proposed and experimental concentrations in unstable condition.

Unstable case	NMSE	FB	COR	FAC2
Proposed	0.01	0.005	0.99	0.96
Previous (unstable)	0.06	0.02	0.96	0.98

It is shown that in Table (8), NMSE and FB close to zero, but COR and FAC2 close to one in unstable condition. So that the proposed model inside a factor of two with observed concentration data and the proposed model achieved 96% from observed data.

Conclusions

Advection-diffusion equation in three-dimensions is solved with Laplace transform and separation method, considering the mixing height is discretizing into N-sub-layers using the form of wind velocity and vertical turbulence in all stables. The proposed concentrations for all stables were calculated by using the analytical proposed model that was estimated. The comparing between the proposed model and observed isotope concentrations of I^{131} and I^{135} which are taken from the second and the first reactors respectively, Egyptian Atomic Energy Authority at Inshas in neutral, stable and unstable conditions. For unstable conditions the proposed concentrations were compared with the first 1st experimental data recorded for radioactive Iodine-135 (I^{135}) of first reactor. While, for stable and neutral conditions the proposed concentrations were compared with the second 2nd experimental data of Iodine I-131 (I^{131}) released from the second research reactor. Comparing between the proposed model and previous work was taking into consideration. It is clear that there is a good perfect result between the proposed model and observed isotope concentrations. In addition, the statistical techniques show that the existence of a factor of two between the proposed model and the concentrations of the observed isotope. Also, it is shown that NMS, FB are close to zero but COR, FAC are close to one. While, the proposed model achieved 99% from observed data in all stabilities.

Figures Caption

Fig. (1) The variation of proposed and observed concentrations of Iodine-131 (I^{131}) via downwind distance in stable condition.

Fig. (2) The relation between the predicted and observed consternations of Iodine-131 (I^{131}) in stable condition.

Fig. (3) The variation of I^{131} concentrations with downwind distance in neutral case.

Fig. (4) The variability of previous and proposed concentrations with observed concentrations of I^{131} in neutral case.

Fig. (5) The proposed and observed isotope concentrations of (I^{135}) via downwind distance in unstable condition.

Fig. (6) The relation between the proposed and observed consternations of Iodine-135 (I^{135}) in unstable condition.

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